1. Introduction

Field seismic data observations and laboratory rock sample measurements demonstrate that anisotropy and viscosity are widely distributed in the subsurface medium (Carcione, 1992; Thomsen, 1986; Best et al., 2007; Zhubayev et al., 2016). Velocity anisotropy is commonly associated in the subsurface medium with aligned structure (Thomsen, 1986; Alkhalifah, 2000). In addition, attenuation anisotropy coexists with velocity anisotropy when seismic waves propagate through thin layers with various velocity and attenuation characteristics or aligned fluid-filled cracks (Liu et al., 2007; Carcione, 2010; Usher et al., 2017; Guo and McMechan, 2017). The amplitude dimming and waveform distortion of seismic waves are generated due to the viscosity and anisotropy properties of the real earth medium. If the unsatisfactory effects of viscosity and anisotropy on seismic wave propagation are ignored during migration imaging, the location of the recorded interfaces will be deviated and the imaging resolution will be reduced (Dutta and Schuster, 2014; Qu et al., 2017). Therefore, it is important to precisely describe the effects of viscosity and anisotropy on seismic wave propagation in order to counteract these negative effects on high-precision imaging.

Viscosity, an anelastic property of the subsurface medium, which has been observed in many field surveys and laboratory measurements, particularly in the strong attenuation regions (e.g., hydrocarbon reservoir), will result in inherent attenuation effects (McDonal et al., 1958; Guo et al., 2016). The intrinsic attenuation characteristics can be expressed by the quality factor \( Q \) that quantifies the energy loss due to subsurface medium absorption at each wavelength (Aki and Richards, 1980; Zhu et al., 2013; Da Silva et al., 2019). In past decades, in order to simulate seismic wave propagation in viscoacoustic media, many viscoacoustic wave equations have been proposed based on the standard linear solid (SLS) model (Emmerich and Korn, 1987; Carcione et al., 1988; SLS) model (Emmerich and Korn, 1987; Carcione et al., 1988;
Robertson et al., 1994; Deng and McMechan, 2007; Zhu et al., 2013) and the constant Q model (Carcione et al., 2002; Carcione, 2010; Zhu and Harris, 2014; Wang et al., 2018, 2020). Viscoacoustic wave equations derived from constant Q theory (Kjartansson, 1979) have naturally separated amplitude dissipation terms and phase dispersion terms, which have attracted a lot of attention. Based on constant Q theory, a fractional Laplacian viscoacoustic wave equation proposed by Zhu and Harris (2014) is widely used for wavefield simulations and Q-compensated RTM in viscoacoustic media (Zhu et al., 2014; Sun et al., 2015; Li et al., 2016). Following that, several high-precision viscoacoustic wave equations have been developed recently (Mu et al., 2021; Yang and Zhu, 2018; Liu and Luo, 2021), which are based on constant Q theory.

Another widely existing characteristic of subsurface media is anisotropy. To describe anisotropy, four independent parameters are used by geophysicists to determine the phase and group velocities of seismic waves in the transversely isotropic (TI) medium (Tsankin, 1996; Fomel, 2004; Li and Stovas, 2021). The propagation of seismic wave in anisotropic media can be precisely described by the multi-parameter dependent anisotropic elastic wave equations (Duveneck and Bakker, 2011; Zhang et al., 2011) and the pure-acoustic wave equations derived from constant 

In addition, although the fractional Laplacian viscoacoustic anisotropic wave equations can be well used for wavefield simulations and Q-compensated RTM, the numerical simulations of these pure-viscoacoustic anisotropic wave equations suffer from SV-wave artifacts. To address this issue, Mu et al. (2022b) and Qiao et al. (2022) independently developed the decoupled pure-viscoacoustic TTI wave equation in media with velocity and attenuation anisotropy. Nevertheless, in media with strong attenuation and anisotropy, the wavefields simulated by these pure-viscoacoustic TTI wave equations are inaccurate. As a result, it is necessary to develop a pure-viscoacoustic TTI wave equation with high accuracy for wavefield simulation in attenuating TTI media.

In this paper, starting from the exact complex-valued phase velocity formula in viscoelastic VTI media, we derive a new pure-viscoacoustic TTI wave equation using the new acoustic approximation that is totally S-wave free. Our new pure-viscoacoustic TTI wave equation can provide more accurate wavefield than the previous wave equations in media with velocity and attenuation anisotropy. The accuracy of the proposed wave equation is confirmed through the theoretical analysis. Then, with the help of the numerical tests, we further verify that the proposed wave equation has higher accuracy in describing qP-wave kinematic properties and attenuation characteristics than the pure-viscoacoustic TTI wave equation proposed by Mu et al. (2022b). In numerical simulations, we develop the hybrid finite-difference and low-rank decomposition (HFDLRD) method to accurately solve our new pure-viscoacoustic TTI wave equation. The numerical test in a simple two-layer shows that the proposed HFDLRD method
parameters. The complex stiffness coefficient $M_{ij}(\omega)$ in Eq. (1) can be written as

$$M_{ij}(\omega) = C_{ij} \cos^2 \left( \frac{\pi \gamma_{ij}}{2} \right) \left( \frac{\omega}{\omega_0} \right)^{2r_v},$$

(2)

where $\omega_0$ denotes the reference angular frequency, $\omega_0 = \text{arctan}(1/Q_0)/\pi$ are dimensionless parameters related to the quality factor and the value of $\gamma_{ij}$ in Eq. (2) is $(0, 0.5)$ for any positive quality factor $Q_0$. The elastic stiffness coefficient $C_{ij}$ can be computed from Thomsen anisotropy parameters $\epsilon$ and $\delta$ (Thomsen, 1986). The $Q$-related Thomsen anisotropy parameters $\epsilon_Q$ and $\delta_Q$ can be used to characterize the anisotropic quality factors $Q_{ij}$ (Zhu and Tsvankin, 2006). Additionally, the frequency-dependent complex-valued velocity can be used to describe the seismic wave propagation in attenuating media. By solving the viscoelastic Christoffel equation in VTI media, the exact complex-valued phase velocity formula in 2D viscoelastic media can be written as

$V_p^2(\theta) = \left( M_{11} \sin^2 \theta + M_{33} \cos^2 \theta + M_{55} + E \right)/2\rho,$

(3)

$V_{sv}^2(\theta) = \left( M_{11} \sin^2 \theta + M_{33} \cos^2 \theta + M_{55} - E \right)/2\rho,$

(4)

where $V_p(\theta)$ and $V_{sv}(\theta)$ denote the P- and SV-wave phase velocity, respectively. $\theta$ and $\rho$ are the phase angle and the density, respectively.

Based on Eq. (3), using the acoustic approximation and other approximations, some pure-viscoacoustic TTI wave equations have been developed recently (Qiao et al., 2022; Mu et al., 2022b). Nevertheless, due to the use of these approximations in deriving the wave equation, the previous pure-viscoacoustic TTI wave equations have low simulation accuracy. Therefore, on the basis of the new acoustic approximation (Xu et al., 2020), we derive a high-precision pure-viscoacoustic TTI wave equation in this study. First, the complex stiffness coefficient $M_{13}$ in Eq. (5) can be approximated using the expression given as follows (Qiao et al., 2019):

$$M_{13} = M_{33}(1 + \delta) - 2M_{55},$$

(6)

Substituting Eq. (6) into Eq. (5), after several mathematical manipulations, Eq. (5) can be expressed as

$$E = \sqrt{\left( M_{33} \cos^2 \theta + M_{11} \sin^2 \theta - M_{55} \right)^2 + 4 \left[ (1 + 2\delta)M_{33}^2 + (M_{11} - (1 + 2\delta)M_{33})M_{55} - M_{11}M_{33} \right] \sin^2 \theta \cos^2 \theta},$$

(7)

where $E$ denotes the angular frequency, $\omega_0$ denotes the reference angular frequency, $\gamma_{ij}$ is $\text{arctan}(1/Q_{ij})/\pi$ are dimensionless parameters related to the quality factor and the value of $\gamma_{ij}$ in Eq. (2) is $(0, 0.5)$ for any positive quality factor $Q_0$. The elastic stiffness coefficient $C_{ij}$ can be computed from Thomsen anisotropy parameters $\epsilon$ and $\delta$ (Thomsen, 1986). The $Q$-related Thomsen anisotropy parameters $\epsilon_Q$ and $\delta_Q$ can be used to characterize the anisotropic quality factors $Q_{ij}$ (Zhu and Tsvankin, 2006). Additionally, the frequency-dependent complex-valued velocity can be used to describe the seismic wave propagation in attenuating media. By solving the viscoelastic Christoffel equation in VTI media, the exact complex-valued phase velocity formula in 2D viscoelastic media can be written as

$V_p^2(\theta) = \left( M_{11} \sin^2 \theta + M_{33} \cos^2 \theta + M_{55} + E \right)/2\rho,$

(3)

$V_{sv}^2(\theta) = \left( M_{11} \sin^2 \theta + M_{33} \cos^2 \theta + M_{55} - E \right)/2\rho,$

(4)

where $V_p(\theta)$ and $V_{sv}(\theta)$ denote the P- and SV-wave phase velocity, respectively. $\theta$ and $\rho$ are the phase angle and the density, respectively.

Based on Eq. (3), using the acoustic approximation and other approximations, some pure-viscoacoustic TTI wave equations have been developed recently (Qiao et al., 2022; Mu et al., 2022b). Nevertheless, due to the use of these approximations in deriving the wave equation, the previous pure-viscoacoustic TTI wave equations have low simulation accuracy. Therefore, on the basis of the new acoustic approximation (Xu et al., 2020), we derive a high-precision pure-viscoacoustic TTI wave equation in this study. First, the complex stiffness coefficient $M_{13}$ in Eq. (5) can be approximated using the expression given as follows (Qiao et al., 2019):

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(7)

where

$E = \sqrt{\left( M_{33} \cos^2 \theta + M_{11} \sin^2 \theta - M_{55} \right)^2 + 4 \left[ (1 + 2\delta)M_{33}^2 + (M_{11} - (1 + 2\delta)M_{33})M_{55} - M_{11}M_{33} \right] \sin^2 \theta \cos^2 \theta},$

(7)

where $E$ denotes the angular frequency, $\omega_0$ denotes the reference angular frequency, $\gamma_{ij}$ is $\text{arctan}(1/Q_{ij})/\pi$ are dimensionless parameters related to the quality factor and the value of $\gamma_{ij}$ in Eq. (2) is $(0, 0.5)$ for any positive quality factor $Q_0$. The elastic stiffness coefficient $C_{ij}$ can be computed from Thomsen anisotropy parameters $\epsilon$ and $\delta$ (Thomsen, 1986). The $Q$-related Thomsen anisotropy parameters $\epsilon_Q$ and $\delta_Q$ can be used to characterize the anisotropic quality factors $Q_{ij}$ (Zhu and Tsvankin, 2006). Additionally, the
By substituting Eqs. (8) and (9) into Eq. (3), and we assume that the density is constant and equals to 1, Eq. (3) can be written as

\[
\begin{align*}
V_p^2(\theta) &= M_{11} \sin^2 \theta + M_{13} \cos^2 \theta \\
&+ \left[ \left( M_{13} - M_{11} M_{33} / M_{33} \right) \sin^2 \theta \cos^2 \theta \right].
\end{align*}
\] (10)

Additionally, we can rewrite Eq. (11) as

\[
\omega^2 = M_{11} k_x^2 + M_{33} k_z^2 + \left( M_{13}^2 / M_{33} - M_{11} \right) k_x^2 k_z^2 \left( k_x^2 + k_z^2 \right) \left( 1 + 2\alpha k_x^2 + 2(1 + \beta) k_z^2 k_x^2 \right).
\] (14)

The term \((i\omega)^{2\gamma_0}\) in Eq. (14) can be converted into fractional Laplacians to reduce the computational memory (Zhu and Harris, 2014), which can be expressed as

\[
(i\omega)^{2\gamma_0} = \omega^{2\gamma_0} k^{2\gamma_0} \cos \left( \pi \gamma_0 \right) + i \omega^{2\gamma_0 - 1} k^{2\gamma_0 - 1} \sin \left( \pi \gamma_0 \right),
\] (15)

where \(k\) is the spatial wavenumber, \(\nu\) denotes the phase velocity at the reference frequency. Note that the phase velocity \(\nu\) is replaced by \(\nu_{11} = \nu \sqrt{(1 + 2\alpha)}, \nu_{33} = \nu, \) and \(\nu_{13} = \nu \sqrt{(1 + 2\beta)^{1/4}}\) in the process of solving \(M_{11}, M_{13},\) and \(M_{33}\) (Qiao et al., 2020), respectively. \(\nu_p\) denotes the P-wave velocity along the vertical symmetry axis at the reference frequency. Using Eq. (15), \(M_{ij}\) can be written as

\[
M_{ij} = \eta_i k^{2\gamma_0} + i \omega \tau_{ij} k^{2\gamma_0 - 1}, \tag{16}
\]

where

\[
\eta_i = C^{\gamma_i+1} \cos^2 \left( \pi \gamma_0 / 2 \right) \omega_0^{2\gamma_0} \cos \left( \pi \gamma_0 \right), \tag{17}
\]

\[
\tau_{ij} = C^{\gamma_i+0.5} \cos^2 \left( \pi \gamma_0 / 2 \right) \omega_0^{2\gamma_0} \sin \left( \pi \gamma_0 \right). \tag{18}
\]

Inserting Eq. (16) into Eq. (14), we derive the pure-viscoacoustic dispersion relation in VTI media:

\[
\omega^2 = \left( \eta_{11} k^{2\gamma_{11}} + \tau_{11} (i\omega) k^{2\gamma_{11} - 1} \right) k_x^2 + \left( \eta_{33} k^{2\gamma_{11}} + \tau_{33} (i\omega) k^{2\gamma_{11} - 1} \right) k_z^2 + \left[ (a_3 k^{2\gamma_{11}} + b_3 (i\omega) k^{2\gamma_{11} - 1} ) - \left( \eta_{11} k^{2\gamma_{11}} + \tau_{11} (i\omega) k^{2\gamma_{11} - 1} \right) \right] k_x^2 k_z^2 \left( k_x^2 + k_z^2 \right) \left( 1 + 2\alpha k_x^2 + 2(1 + \beta) k_z^2 k_x^2 \right)
\] (19)

In VTI media, the relation between phase velocity, frequency and wavenumber is given as shown below (Zhan et al., 2012):

\[
\sin \theta = \frac{V_p(\theta) k_x}{\omega}, \quad \cos \theta = \frac{V_p(\theta) k_z}{\omega}, \tag{13}
\]

where \(k_x, k_z\) denote \(x, z\) direction wavenumber, respectively. Substituting Eq. (13) into Eq. (12), Eq. (12) can be written as
\[
\frac{\partial^2 p}{\partial t^2} = \left( \eta_{11} \left( -\nabla^2 \right)^\gamma_{11} + \tau_{11} \frac{\partial}{\partial t} \left( -\nabla^2 \right)^{\gamma_{11}-0.5} \right) \frac{\partial^2 p}{\partial x^2} + \left( \eta_{33} \left( -\nabla^2 \right)^{\gamma_{33}} + \tau_{33} \frac{\partial}{\partial t} \left( -\nabla^2 \right)^{\gamma_{33}-0.5} \right) \frac{\partial^2 p}{\partial z^2},
\]

where \( f \) is source function. We notice that Eq. (20) is equivalent to the time-space domain anisotropic pure P-wave equation given by Liang et al. (2023) when \( Q_p \to \infty \).

In TTI media, the pure-viscoacoustic TTI wave equation can be deduced from Eq. (20) through coordinate rotation. The wave-number relationship between VTI and TTI media (Zhan et al., 2012) can be expressed as

\[
\hat{k}_x = \cos \phi \kappa_x - \sin \phi \kappa_z, \quad \hat{k}_z = \sin \phi \kappa_x + \cos \phi \kappa_z.
\]  

(21)

\[
\frac{\partial^2 p}{\partial t^2} = \left( \eta_{11} \left( -\nabla^2 \right)^\gamma_{11} + \tau_{11} \frac{\partial}{\partial t} \left( -\nabla^2 \right)^{\gamma_{11}-0.5} \right) \left( \cos^2 \phi \frac{\partial^2}{\partial x^2} + \sin^2 \phi \frac{\partial^2}{\partial z^2} - \sin (2 \phi) \frac{\partial^2}{\partial x \partial z} \right) p +
\]

\[
\left( \eta_{33} \left( -\nabla^2 \right)^{\gamma_{33}} + \tau_{33} \frac{\partial}{\partial t} \left( -\nabla^2 \right)^{\gamma_{33}-0.5} \right) \left( \sin^2 \phi \frac{\partial^2}{\partial x^2} + \cos^2 \phi \frac{\partial^2}{\partial z^2} + \sin (2 \phi) \frac{\partial^2}{\partial x \partial z} \right) p +
\]

\[
\left[ \left( a_3 \left( -\nabla^2 \right)^{\lambda_{31}^1} + b_3 \frac{\partial}{\partial t} \left( -\nabla^2 \right)^{\lambda_{31}^1-0.5} \right) - \left( \eta_{11} \left( -\nabla^2 \right)^{\gamma_{11}} + \tau_{11} \frac{\partial}{\partial t} \left( -\nabla^2 \right)^{\gamma_{11}-0.5} \right) \right] \left[ \frac{\partial^4}{\partial x^2 \partial^2} + \frac{\partial^4}{\partial z^2 \partial^2} \right] p + f.
\]  

(24)

where

\[
S_t = \begin{pmatrix}
\cos^2 \phi \sin^2 \phi \left( \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial z^4} \right) + (\cos^4 \phi + \sin^4 \phi - \sin (2 \phi)) \frac{\partial^4}{\partial x^2 \partial^2} \\
+ \cos (2 \phi) \sin (2 \phi) \left( \frac{\partial^4}{\partial x^2 \partial^2} - \cos (2 \phi) \sin (2 \phi) \frac{\partial^4}{\partial x \partial z \partial^2} \right)
\end{pmatrix}.
\]

(25)

where \( \phi \) denotes the dip angle of the symmetry axis. Replacing \( k_x \) and \( k_z \) in Eq. (19) with \( \hat{k}_x \) and \( \hat{k}_z \), Eq. (20) can be written as

\[
\omega^2 = \left( \eta_{11} k_x^{2 \gamma_{11}} + \tau_{11} (i \omega) k_x^{2 \gamma_{11}-1} \right) \left( \cos^2 \phi k_x^2 + \sin^2 \phi k_z^2 - \sin (2 \phi) k_x k_z \right) + \left( \eta_{33} k_z^{2 \gamma_{33}} + \tau_{33} (i \omega) k_z^{2 \gamma_{33}-1} \right) \left( \sin^2 \phi k_x^2 + \cos^2 \phi k_z^2 + \sin (2 \phi) k_x k_z \right) + \left[ a_3 k_x^{2 \lambda_{31}^1} + b_3 (i \omega) k_x^{2 \lambda_{31}^1-1} \right] - \left( \eta_{11} k_x^{2 \gamma_{11}} + \tau_{11} (i \omega) k_x^{2 \gamma_{11}-1} \right) S_k \left( k_x^2 + k_z^2 \right).
\]  

(22)
It is notable that Eq. (24) contains decoupled phase dispersion terms and amplitude attenuation terms. In the above equations, the dispersion effects are dominated by a term containing \((-\nabla^2)^{\gamma_1}\), the amplitude attenuation effects are dominated by a term containing \(\frac{\alpha}{\nabla^2} (\nabla^2)^{\gamma_1-0.5} \partial_t\). If we only consider phase dispersion effects, the dispersion-dominated wave equation can be given as

\[
\frac{\partial^2 p}{\partial t^2} = \eta_{11} (-\nabla^2)^{\gamma_1} \left( \cos^2 \phi \frac{\partial^2}{\partial x^2} + \sin^2 \phi \frac{\partial^2}{\partial z^2} - \sin(2\phi) \frac{\partial^2}{\partial x \partial z} \right) p + \\
\eta_{33} (-\nabla^2)^{\gamma_1} \left( \sin^2 \phi \frac{\partial^2}{\partial x^2} + \cos^2 \phi \frac{\partial^2}{\partial z^2} + \sin(2\phi) \frac{\partial^2}{\partial x \partial z} \right) p + \\
\left( \eta_3 (-\nabla^2)^{\lambda_{11}} - \eta_{11} (-\nabla^2)^{\gamma_1} \right) \xi \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right) p + f.
\]

(26)

Similarly, the dissipation-dominated wave equation can be written as

\[
\frac{\partial^2 p}{\partial t^2} = \left( c_{11} + \tau_{11} \frac{\partial}{\partial t} \right) (-\nabla^2)^{\gamma_1-0.5} \left( \cos^2 \phi \frac{\partial^2}{\partial x^2} + \sin^2 \phi \frac{\partial^2}{\partial z^2} - \sin(2\phi) \frac{\partial^2}{\partial x \partial z} \right) p + \\
\left( c_{33} + \tau_{33} \frac{\partial}{\partial t} \right) (-\nabla^2)^{\gamma_1-0.5} \left( \sin^2 \phi \frac{\partial^2}{\partial x^2} + \cos^2 \phi \frac{\partial^2}{\partial z^2} + \sin(2\phi) \frac{\partial^2}{\partial x \partial z} \right) p + \\
\left[ \left( c_{13} + \tau_3 \frac{\partial}{\partial t} \right) (-\nabla^2)^{\lambda_{11}-0.5} - \left( c_{11} + \tau_{11} \frac{\partial}{\partial t} \right) (-\nabla^2)^{\gamma_1-0.5} \right] \xi \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right) p + f.
\]

(27)

2.2. Theoretical accuracy analysis of the proposed pure-viscoacoustic TTI wave equation

In this section, several theoretical analysis experiments are performed to investigate the accuracy of the newly proposed wave equation. The comparisons between the exact complex-valued phase velocity formula for P-wave in viscoelastic TTI media (Eq. (A-1)), the formula proposed by Mu et al. (2022b), and our formula are generated for accuracy analysis. First, we plot the phase velocity curves for different parameters shown in Fig. 1, the model parameters are given in Table 1. Note that the phase velocity can be generated using Eq. (A-3). The source dominant frequency is the same as the reference frequency, which is 30 Hz.

Fig. 1 shows that the phase velocity curves of the proposed formula (Eq. (24)) are in better match with the phase velocity curves of exact formula than that of the formula given by Mu et al. (2022b). With velocity anisotropy strengthening, the newly derived formula is more accurate than the formula of Mu et al. (2022b), as shown in Fig. 1(b). These results suggest that the newly derived formula has higher accuracy than the formula derived by Mu et al. (2022b).

Furthermore, we generate the maximum relative error to
investigate the accuracy of the newly derived wave equation. The maximum relative error function given by Mu et al. (2020) can be reformulated as

\[
E_R(x, y) = \max \left( \frac{|V_e(x, y, \theta) - V_a(x, y, \theta)|}{V_a(x, y, \theta)} \right), \quad \theta \in \left(0, \frac{\pi}{2}\right),
\]

where \(V_e(x, y, \theta)\) denotes the exact formula of P-wave in viscoelastic anisotropic media; \(V_a(x, y, \theta)\) represents the approximate P-wave formula in viscoacoustic anisotropic media; \(x\) and \(y\) denote the variable we are studying. Fig. 2(a) (b) shows the maximum relative error of the phase velocity of different approximate formulas. Fig. 2(c) (d) shows the maximum relative error of the quality of different approximate formulas. The phase velocity and quality factor can be generated using Eq. (A-3) and Eq. (A-4), respectively. In Fig. 2(a) (b), the proposed formula has more area with relative phase velocity error less than 1%, in comparison to the formula proposed by Mu et al. (2022b). This result suggests that the newly derived equation has higher accuracy in representing velocity

<table>
<thead>
<tr>
<th>Model</th>
<th>Model parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>(v_p, \text{m/s})</td>
<td>(v_s, \text{m/s})</td>
</tr>
<tr>
<td>Model A</td>
<td>3000</td>
</tr>
<tr>
<td>Model B</td>
<td>3000</td>
</tr>
<tr>
<td>Model C</td>
<td>3000</td>
</tr>
<tr>
<td>Model D</td>
<td>3000</td>
</tr>
</tbody>
</table>

Fig. 2. The maximum relative error of the phase velocity and quality factor of the different approximation formulas. (a) and (b) are the maximum relative error of the phase velocity; (c) and (d) are the maximum relative error of the quality factor. The attenuation anisotropy parameters of (a) and (b) are \(\epsilon_Q = -0.4, \delta_Q = -0.3\). The velocity anisotropy parameters of (c) and (d) are \(\epsilon = 0.15, \delta = 0.1\). The P-wave model parameters are \(v_p = 3000 \text{ m/s}\), \(Q_p = 30\), and the parameters \(v_s = 1800 \text{ m/s}\) and \(Q_s = 20\) are for viscoelastic anisotropic media.
anisotropy than the formula given by Mu et al. (2022b). Moreover, from Fig. 2(c)–(d), one can clearly see that the maximum relative error of quality factor of the proposed formula is obvious smaller than the formula proposed by Mu et al. (2022b), which demonstrates the proposed formula with higher precision in representing attenuation anisotropy. From the above results, one can conclude that the newly derived equation has higher accuracy than the formula developed by Mu et al. (2022b) in describing velocity anisotropy and attenuation anisotropy.

3. Numerical implementations

In this section, we develop the HFDLRD method to accurately solve our new wave equation. The variable-order fractional Laplacian and mixed-domain operator $S_2$ are solved by the low-rank decomposition method (Sun et al., 2016), and the other partial derivatives are solved by the finite-difference method. The low-rank decomposition method was developed by Fomel et al. (2013), which can be expressed as

$$W(x, k) = \sum_{m=1}^{M} W_1(x, k_m) \sum_{n=1}^{N} a_{mn} W_2(x_n, k). \quad (29)$$

where $W_1(x, k_m)$ and $W_2(x_n, k)$ are the submatrices of $W(x, k)$, which are related to the wavenumbers and spatial locations, respectively. The coefficient $a_{mn}$ is the connection between $W_1(x, k_m)$ and $W_2(x_n, k)$, and $m$ and $n$ is the rank of the matrices. In addition, the coefficient $a_{mn}$ can be determined using expression $a_{mn} = W(x_n, k_m)$, where $\dagger$ denotes the pseudo-inverse (see more detail in Fomel et al., 2013). Based on the HFDLRD strategy, Eq. (24) can be rewritten as

$$\frac{\partial^2 p}{\partial t^2} = (a_{11} q_1 + b_{11} q_2) \left( \cos^2 \varphi \frac{\partial^2}{\partial x^2} + \sin^2 \varphi \frac{\partial^2}{\partial z^2} - \sin 2 \varphi \frac{\partial^2}{\partial x \partial z} \right) +$$

$$(a_{33} q_3 + b_{33} q_4) \left( \sin^2 \varphi \frac{\partial^2}{\partial x^2} + \cos^2 \varphi \frac{\partial^2}{\partial z^2} + \sin 2 \varphi \frac{\partial^2}{\partial x \partial z} \right) +$$

$$(a_{35} q_5 + b_{36} q_6 - (a_{11} q_7 + b_{11} q_8)) \left( \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) + f. \quad (30)$$

where $q_1 = \mathcal{F}^{-1}(k^{7/3} \mathcal{F}(p))$, $q_2 = \mathcal{F}^{-1}(k^{2/3} \mathcal{F}(\partial p/\partial t))$, $q_3 = \mathcal{F}^{-1}(k^{7/3} \mathcal{F}(p))$, $q_4 = \mathcal{F}^{-1}(k^{2/3} \mathcal{F}(\partial p/\partial t))$, $q_5 = \mathcal{F}^{-1}(S_2 \mathcal{F}(p))$, $q_6 = \mathcal{F}^{-1}(S_2 \mathcal{F}(\partial p/\partial t))$, $q_7 = \mathcal{F}^{-1}(S_2 \mathcal{F}(p))$, $q_8 = \mathcal{F}^{-1}(S_2 \mathcal{F}(\partial p/\partial t))$, $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the forward fast Fourier transforms (FFTs) and inverse FFTs, respectively. Based on Eq. (28), $q_1$ can be solved by the low-rank decomposition as follows:

$$\mathcal{F}^{-1} \left( \left( -\mathcal{F} \right)^{7/3} \mathcal{F}(p) \right) = \sum_{m=1}^{M} W_1(x, k_m) \left( \sum_{n=1}^{N} a_{mn} \mathcal{F}^{-1} (W_2(x_n, k) \mathcal{F}(p)) \right). \quad (31)$$

Correspondingly, $q_5 - q_8$ also can be solved by the low-rank decomposition method. To solve Eq. (31), we can use a small rank of low-rank decomposition to meet the accuracy requirement well, because Eq. (31) is independent by the velocity model (Yan and Liu, 2016; Zhang et al., 2019). Here, if the ranks of the low-rank decomposition in solving $q_1 - q_4$ and $q_5 - q_8$ are defined as $N_1$ and $N_2$ respectively, then, the number of forward FFTs and inverse FFTs are 2 and $4N_1 + 4N_2$ respectively.

4. Numerical examples

In this section, we use several homogeneous and heterogeneous models to illustrate the accuracy and stability of our proposed pure-viscoacoustic TTI wave equation (Eq. (24)). The coupled pseudo-viscoacoustic TTI wave equation derived by Qiao et al. (2020) is used as the reference to evaluate the accuracy of the newly derived equation, due to the wavefield simulated by the coupled pseudo-viscoacoustic TTI wave equation preserves accurately the kinematic features (Mu et al., 2022b). For numerical implementation, all the numerical examples used in this section are solved by the HFDLRD method. In addition, the density is uniformly defined as $\rho=1$ in all numerical examples. A simple two-layer model is employed to investigate the accuracy of the proposed HFDLRD method. The boundary reflections are attenuated by using the sponge absorbing boundary (Cerjan et al., 1985).

4.1. A homogeneous model

4.1.1. Accuracy analysis of the proposed pure-viscoacoustic TTI wave equation

In this case, we build a homogeneous model to perform wavefield simulation to illustrate the accuracy of our new pure-viscoacoustic TTI wave equation. The homogeneous model is built of $401 \times 401$ grids and discretized with grid spacing of 10 m × 10 m. A Ricker wavelet with the dominant frequency of 25 Hz is injected at the central of model; the time step is 0.001 s; the reference frequency is 25 Hz. Fig. 3 shows the snapshots at 0.5 s for different attenuating VTI models simulated by the coupled pseudo-viscoacoustic TTI wave equation of Qiao et al. (2020), the pure-viscoacoustic TTI wave equation derived by Mu et al. (2022b), and our new wave equation, respectively. The ranks of the low-rank decomposition method are $N_1 = 1$ and $N_2 = 1$.

From Fig. 3, one can observe that the wavefields generated by the coupled pseudo-viscoacoustic TTI wave equation produce the S-wave artifacts, while the wavefields generated by the pure-viscoacoustic TTI wave equation are free of S-wave artifacts. This result suggests that the wavefields simulated by pure-viscoacoustic TTI wave equation are noise-free. Additionally, Figs. (4) and (5) show the wavefield snapshots comparison in a wiggle format. Fig. 4(a) is the superposition of Fig. 3(e) (red dashed line) and Fig. 3(d) (black solid line). Fig. 4(b) is the superposition of Fig. 3(f) (red dashed line) and Fig. 3(d) (black solid line). Similarly, Fig. 5(a) is the superposition of Fig. 3(h) (red dashed line) and Fig. 3(g) (black solid line). Fig. 5(b) is the superposition of Fig. 3(i) (red dashed line) and Fig. 3(g) (black solid line). Given that the attenuation param-
agreement with the reference traces (black line) than the wavefields simulated by the pure-viscoacoustic TTI wave equation derived by Mu et al. (2022b), as pointed out by the black arrows. This result illustrates that the newly derived wave equation has higher accuracy in describing velocity anisotropy than the pure-viscoacoustic TTI wave equation proposed by Mu et al. (2022b). Similar with Fig. 4, the wavefields calculated by the proposed wave equation are closer to the reference wavefields than that of the pure-viscoacoustic TTI wave equation given by Mu et al. (2022b). This finding demonstrates that the wavefields simulated by our new wave equation are more accurate than the pure-viscoacoustic TTI wave equation derived by Mu et al. (2022b) in representing the attenuation anisotropy characteristics. Therefore, according to the above results, we can conclude that the newly derived wave equation has higher accuracy than the pure-viscoacoustic TTI wave equation proposed by Mu et al. (2022b) in describing velocity anisotropy and attenuation anisotropy. These numerical results also are consistent with the theoretical analysis displayed in Figs. 1 and 2.

4.1.2. Modeling of the decoupled amplitude attenuation and phase dispersion effects

To demonstrate the effects of decoupled amplitude attenuation and phase dispersion of our pure-viscoacoustic TTI wave equation, we build a homogenous model with grid points of 401 × 401 and a spacing of 10 m. The model parameters are $v_p = 3000$ m/s, $\varepsilon = 0.35$, $\delta = 0.05$, $\varphi = 0$, $Q_p = 30$. The attenuation anisotropy parameters are: (a)–(c) $\varepsilon_Q = 0.7$, $\delta_Q = 0.15$; (d)–(f) $\varepsilon_Q = 0$, $\delta_Q = 0$. (g) is the difference between (a) and (d), (h) is the difference between (b) and (e), and (i) is the difference between (c) and (f). The first column to the third column are the wavefields simulated by the coupled pseudo-viscoacoustic TTI wave equation derived by Qiao et al. (2020), the pure-viscoacoustic TTI wave equation proposed by Mu et al. (2022b), and our new pure-viscoacoustic TTI wave equation, respectively.
amplitude decrease and phase delay. Fig. 6(b) shows wavefield snapshots at 0.5 s simulated by pure-viscoacoustic TTI wave equation with different quality factors, which illustrates that the lower the quality factor, amplitude attenuation and phase delay are more serious. The above results suggest that our pure-viscoacoustic anisotropic wave equation can achieve decoupled amplitude attenuation and phase dispersion wavefield simulations, which facilitate the realization of Q-compensated RTM in attenuating anisotropic media.

4.2. A simple two-layer model

In this case, we use a simple two-layer model to investigate the accuracy of the HFDLRD method in solving the pure-viscoacoustic TTI wave equation. Note that the pure-viscoacoustic TTI wave equation proposed by Mu et al. (2022b) is used for generating wavefields in this case. The model is discretized with 401 x 401 grid points uniformly along the vertical and horizontal directions with a spacing of 10 m. The source is a Ricker wavelet with a peak frequency of 25 Hz, which is placed at the central of model. The time step is 0.001 s and the reference frequency is 25 Hz. The reference wavefield is calculated by the blocked computing method (Li et al., 2016). We use the HFDP5 method based on the second-order Taylor series expansion approximation of Zhang et al. (2020b) to generate wavefields for comparison. The ranks of the low-rank decomposition method are $N_1 = 2$ and $N_2 = 2$. Wavefield snapshots at 0.9 s for
the simple two-layer model calculated by the blocking method, the HFDPS method, and the HFDLRD method are shown in Fig. 7(a)–(c), respectively. The corresponding wavefield differences are shown in Fig. 7(d)–(e). From Fig. 7, one can see that the differences (Fig. 7(e)) between Fig. 7(a) and Fig. 7(c) are almost zero, while the differences (Fig. 7(d)) between Fig. 7(a) and Fig. 7(b) are apparent. The above findings suggest that HFDLRD method outperforms HFDPS method in terms of accuracy of wavefield simulation in attenuating anisotropic media, especially in the case of strong attenuation (Q < 10).

4.3. A modified Hess partial model

The modified complex Hess partial model is further used to verify the accuracy of the newly proposed equation simulated in complex media, and the model parameters are shown in Fig. 8. The model with size of 6 km × 3 km and discretized with grid spacing of 10 m × 10 m. A Ricker wavelet with a peak frequency of 20 Hz is located at (3000 m, 10 m). The time step is 0.001 s, the reference frequency is 20 Hz. The wavefield snapshots at 1.15 s generated by
different equations are shown in Fig. 9. The ranks of the low-rank decomposition method are $N_1 = 2$ and $N_2 = 2$. In Fig. 9(a), there are shear wave artifacts in the wavefields (as shown in the black dashed rectangular box), while the wavefields simulated by pure-viscoacoustic TTI wave equation doesn’t contain (as shown in Fig. 9(b)–(c)), which illustrates that pure-viscoacoustic TTI wave equation can obtain cleaner results than coupled pseudo-viscoacoustic TTI wave equation. For better comparison, the extracted traces from Fig. 9 are shown in Fig. 10. Fig. 10 shows that the traces extracted from Fig. 9(b) (red dashed line) are closer to the traces extracted from Fig. 9(c) (black solid line) than the traces extracted from Fig. 9(c) (pink dashed line). From the above results, one can conclude that the newly derived pure-viscoacoustic TTI wave equation can achieve more accurate numerical modeling results than that of the wave equation derived by Mu et al. (2022b) in complex attenuating anisotropic media.

4.4. A modified BP 2007 model

To demonstrate the stability of our pure-viscoacoustic TTI wave equation in complex model, the modified BP 2007 model is employed to perform the wavefield simulation. The model is discretized by $701 \times 451$ grid points with a uniform vertical and horizontal space step of 15 m, the model parameter as displayed in Fig. 11. A Ricker wavelet with the dominant frequency of 20 Hz is located at $(5257.5\,\text{m}, 10\,\text{m})$; the time step is 0.001 s; the reference frequency is 20 Hz. The ranks of the low-rank decomposition method are $N_1 = 2$ and $N_2 = 3$. Fig. 12(a)–(b) shows the wavefields simulated by the coupled pseudo-viscoacoustic TTI wave equation of Qiao et al. (2020) at 1.5 and 2.5 s, respectively. Fig. 13(a)–(d) shows the wavefields generated by our pure-viscoacoustic TTI wave equation at 1.5, 2.5, 3.0 and 3.5 s, respectively. It can be distinctly seen that Fig. 12(a) produces the numerical instability (as pointed...
out by the black arrows), and the strong numerical instability is appeared in Fig. 12(b). On the contrast, the wave fields generated by our pure-viscoacoustic TTI wave equation can remain stable, as displayed in Fig. 13. The above results demonstrate that our pure-viscoacoustic TTI wave equation can obtain a more stable wave-field in complex media in comparison to the coupled pseudo-viscoacoustic TTI wave equations.

5. Discussion

In this paper, a new pure-viscoacoustic anisotropic wave equation is derived from the exact complex-valued dispersion relation in viscoelastic VTI media and we develop the HFDLRD method to accurately solve the proposed wave equation. The newly derived equation has higher accuracy than the previous wave equations, while also has drawback that requires huge computational resources. This is because the existence of the operator $S_{\delta}$ in Eq. (24). To improve the computational efficiency, we provide a simplified wave equation as follows:

We rewrite Eq. (11) as

$$V_p^2(\theta) = M_{11} \sin^2 \theta + M_{33} \cos^2 \theta$$

$$+ \frac{\left[ M_{13}^2 / M_{33} - M_{11} \right] \sin^2 \theta \cos^2 \theta}{(1 + \epsilon) + \left( \epsilon \sin^2 \theta - 1 \right) - 0.5(\epsilon - \delta) \sin^2 2\theta}.$$  \hspace{1cm} (32)

given that $0.5(\epsilon - \delta) \sin^2 2\theta \leq 0.075$ for $\epsilon \in (0, 0.4)$, $\theta \in (0, 2\pi)$ and $\delta \in (-0.2, 0.4)$, we can make an assumption that $0.5(\epsilon - \delta) \sin^2 2\theta \approx 0$. In addition, we make an assumption that $\sin^2 \theta = 0.5$, because the average of $\sin^2 \theta$ is 0.5 for $\theta \in (0, 2\pi)$ (Huang et al., 2023). Therefore, Eq. (32) can be approximated as follows:

$$V_p^2(\theta) = M_{11} \sin^2 \theta + M_{33} \cos^2 \theta$$

$$\quad + \frac{\left[ M_{13}^2 / M_{33} - M_{11} \right] \sin^2 \theta \cos^2 \theta}{1 + \epsilon}.$$ \hspace{1cm} (33)
Using Eqs. (13) and (15), Eq. (33) can be formulated as

$$\omega^2 = \left( \eta_{11} k^2 \gamma_{11} + \tau_{11}(i\omega) k^2 \gamma_{11}^{-1} \right) k_x^2 + \left( \eta_{33} k^2 \gamma_{33} + \tau_{33}(i\omega) k^2 \gamma_{33}^{-1} \right) k_z^2 \frac{\left( a_3 k^2 \gamma_{33} + b_3(i\omega) k^2 \gamma_{33}^{-1} \right) k_y^2 k_z^2}{1 + \varepsilon}. \tag{34}$$

Eq. (34) is the approximate dispersion relation of P-wave in VTI media. Based on Eq. (34), finally, the simplified time-space domain pure-viscoacoustic TTI wave equation is given as

$$\frac{\partial^2 p}{\partial t^2} = \left( \eta_{11} \left( -\nabla^2 \right)^{\gamma_{11}} + \tau_{11} \frac{\partial}{\partial t} \left( -\nabla^2 \right)^{\gamma_{11}-0.5} \right) \left( \cos^2 \varphi \frac{\partial^2}{\partial x^2} + \sin^2 \varphi \frac{\partial^2}{\partial z^2} - \sin(2\varphi) \frac{\partial^2}{\partial x \partial z} \right) p +$$

$$\left( \eta_{33} \left( -\nabla^2 \right)^{\gamma_{33}} + \tau_{33} \frac{\partial}{\partial t} \left( -\nabla^2 \right)^{\gamma_{33}-0.5} \right) \left( \sin^2 \varphi \frac{\partial^2}{\partial x^2} + \cos^2 \varphi \frac{\partial^2}{\partial z^2} + \sin(2\varphi) k_y k_z \right) p +$$

$$\left( a_3 \left( -\nabla^2 \right)^{\gamma_{33}} + b_3 \frac{\partial}{\partial t} \left( -\nabla^2 \right)^{\gamma_{33}-0.5} \right) \left( \cos^2 \varphi \sin^2 \varphi \left( \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial z^4} + \frac{\partial^4}{\partial x^2 \partial z^2} \right) + \cos^4 \varphi + \sin^4 \varphi - \sin^2(2\varphi) \frac{\partial^4}{\partial x^2 \partial z^2} \right) p$$

$$+ \cos(2\varphi) \sin(2\varphi) \frac{\partial^4}{\partial x^3 \partial z} - \cos(2\varphi) \sin(2\varphi) \frac{\partial^4}{\partial x \partial z^3} \right) p + f. \tag{35}$$

To demonstrate the accuracy of the simplified pure-viscoacoustic TTI wave equation (Eq. (35)), we generate the wavefield snapshots at 0.5 s using the different wave equations. The model parameters are the same as those in Fig. 3(a). The Ricker wavelet with peak frequency of 25 Hz is located at the central of the model, the time step is 0.001 s, the reference frequency is 25 Hz.

![Wavefield snapshots simulated by our pure-viscoacoustic TTI wave equation at t = 1.5 s (a), t = 2.5 s (b), t = 3.0 s (c), t = 3.5 s (d).](image-url)
Fig. 14. Wavefield snapshots at 0.5 s generated by different wave equation. (a) The coupled pseudo-viscoacoustic TTI wave equation derived by Qiao et al. (2020). (b) The pure-viscoacoustic TTI wave equation proposed by Mu et al. (2022b). (c) The proposed pure-viscoacoustic TTI wave equation (Eq. (24)). (d) The proposed simplified pure-viscoacoustic TTI wave equation (Eq. (35)).

Fig. 15. Comparisons of wavefield snapshots from Fig. 14(a)–(d) in wiggle forms. (a) The wavefields generated by the pure-viscoacoustic TTI wave equation derived by Mu et al. (2022b) (Fig. 14(b)) (red dashed lines) and the reference wavefield (black solid lines). (b) The wavefields generated by the proposed pure-viscoacoustic TTI wave equation (Fig. 14(c)) (red dashed lines) and the reference wavefield (black solid lines). (c) The wavefields generated by the proposed simplified pure-viscoacoustic TTI wave equation (Fig. 14(d)) (red dashed lines) and the reference wavefield (black solid lines).
Fig. 14 shows the wavefield snapshots at 0.5 s generated by the different wave equations. The ranks of the low-rank decomposition method are $N_1 = 1$ and $N_2 = 1$. Fig. 15 are the comparisons of wavefield snapshots in a wiggle form. As pointed out by the black arrows in Fig. 15, one can notice that the wavefield snapshots generated by the proposed wave equation (Fig. 15(b)) and the proposed simplified wave equation (Fig. 15(c)) are in better match with the reference wavefield, in comparison with the previous wave equation (Fig. 15(a)). This result suggests that both the proposed wave equation and the proposed simplified wave equation can accurately simulate the kinematic characteristic of P-wave in attenuating anisotropic media. Additionally, compared to the proposed wave Eq. (24), the proposed simplified wave Eq. (35) requires less computational cost because it has concise expression. However, it also should be noted that the proposed simplified wave equation has made more approximation than the proposed wave equation (Eq. (24)) in the derivation of the wave equation. This leads to the fact that the accuracy of the proposed simplified wave equation is lower than the proposed wave equation. To further fairly evaluate the accuracy of the proposed simplified wave equation, we provide the maximum relative error of the phase velocity and quality factor of the proposed simplified wave equation, as shown in Fig. 16. By comparing Fig. 16 with Fig. 2, we can observe that the accuracy of the proposed simplified wave equation is higher than the previous wave equations, while it is lower than the proposed wave equation (Eq. (24)). As a result, from the above analysis, we can see that the two types of proposed pure-viscoacoustic anisotropic equations (i.e., Eqs. (24) and (35)) have their features, and we can choose between them based on our requirements.

Second, the numerical stability of wavefield simulation in attenuating anisotropic media also has attracted a lot of interesting (Mu et al., 2022b). We notice that although the pure-viscoacoustic TTI wave equation is more stable than the coupled pseudo-viscoacoustic TTI wave equation, numerical instability still occurs when simulating in some regions with rapidly changing tilt angle (Duveneck and Bakker, 2011; Yan and Liu, 2016). This numerical instability can be attributed to the fact that the acoustic approximation is used and all spatial derivatives in the direction of the anisotropic symmetry axis are neglected in the derivation of the wave equation (Duveneck and Bakker, 2011; Mu et al., 2022b). To solve this problem, several measures (e.g., low-pass filter) can be used to address the instability to stabilize the computation results when simulating in complex media with a sharp tilt angle (Mu et al., 2020). Nevertheless, the kinematic and dynamic accuracy of the wavefield may be affected by these approaches to some extent. Therefore, the development of an accurate and suitable method to address these instabilities is our future work.

6. Conclusion

Based on the exact complex-valued dispersion relation in viscoelastic VTI media, we derive a new pure-viscoacoustic TTI wave equation in media with velocity anisotropy and attenuation anisotropy. This equation contains decoupled phase dispersion and amplitude dissipation terms, which makes it convenient to realize Q-compensated RTM. Theoretical analysis and numerical experiments demonstrate that our pure-viscoacoustic TTI wave equation has higher accuracy than the previous pure-viscoacoustic TTI wave equations in modeling seismic wave propagation in attenuating anisotropic media. For numerical simulation, we develop the HFDLRD method to solve the proposed pure-viscoacoustic TTI wave equation. Numerical test of the simple two-layer shows that the HFDLRD method can accurately calculate the seismic wave propagation in attenuating anisotropic media with strong attenuation. The newly derived pure-viscoacoustic TTI wave equation and the proposed numerical simulation method can be used as forward engines for viscoacoustic anisotropic RTM and FWI in attenuating anisotropic media.

Declaration of competing interest

All authors certify that they have no affiliations with or involvement in any organization or entity with any financial interest or non-financial interest in the subject matter or materials discussed in this manuscript.

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Appendix A

The exact and approximate TTI complex-valued P-wave phase velocity formulas

Starting from Eq. (3), we rotate the symmetry axis and the exact complex-valued phase velocity formula for P-wave in the visco-elastic TTI media can be written as

\[
V(p, \phi) = \left[ \frac{1}{2} \left( M_{11} \sin^2(\phi - \theta) + M_{33} \cos^2(\phi - \theta) + M_{55} + E \right) \right]^{1/2},
\]

\[
E = \left( (M_{33} - M_{55}) \cos^2(\phi - \theta) + (M_{55} - M_{11}) \sin^2(\phi - \theta) \right)^2 + 4(M_{13} - M_{55})^2 \sin^2(\theta) \cos^2(\phi - \theta)
\]

(A-1)

Similarly, the proposed approximate complex-valued phase velocity formula in TTI media can be expressed as

\[
V^2(p, \phi) = M_{11} \sin^2(\phi - \theta) + M_{33} \cos^2(\phi - \theta) + \frac{[M_{13}/M_{33} - M_{11}] \sin^2(\phi - \theta) \cos^2(\phi - \theta)}{\left( (1 + 2\epsilon) \sin^2(\phi - \theta) + \cos^2(\phi - \theta) - 2(\epsilon - \delta) \sin^2(\phi - \theta) \cos^2(\phi - \theta) \right)}.
\]

(A-2)

According to Eq. (36), Qiao et al. (2020) also give the expressions of the directionally dependent phase velocity, quality factor and attenuation coefficient of P-wave, which can be written as follows:

\[
\hat{v}_p(p) = \left( \text{Re} \left( \frac{1}{V_p} \right) \right)^{-1},
\]

\[
\hat{Q}_p(p) = \frac{\text{Re} \left( \frac{V_p^2}{V_p} \right)}{\text{Im} \left( \frac{V_p^2}{V_p} \right)},
\]

\[
\hat{a}_p(p) = -\omega \text{Im} \left( \frac{1}{V_p} \right).
\]

(A-3)

(A-4)

(A-5)

Appendix B

The HFDLRD method for solving the coupled pseudo-viscoacoustic TTI wave equation

Qiao et al. (2020) proposed a coupled pseudo-viscoacoustic TTI wave equation, which can be rewritten as

\[
\frac{\partial^2 \sigma_{xx}}{\partial t^2} = \left( \eta_{11} \left( -\nabla^2 \right)^{\gamma_{11}} + \tau_{11} \frac{\partial}{\partial t} \left( -\nabla^2 \right)^{\gamma_{11}/2 - 0.5} \right) \left( \cos^2 \phi \frac{\partial^2}{\partial x^2} + \sin^2 \phi \frac{\partial^2}{\partial z^2} - \sin 2 \theta \frac{\partial^2}{\partial x \partial z} \right) \left( \sigma_{xx} \right)
\]

\[
+ \left( \eta_{13} \left( -\nabla^2 \right)^{\gamma_{13}} + \tau_{13} \frac{\partial}{\partial t} \left( -\nabla^2 \right)^{\gamma_{13}/2 - 0.5} \right) \left( \sin^2 \phi \frac{\partial^2}{\partial x^2} + \cos^2 \phi \frac{\partial^2}{\partial z^2} + \sin 2 \phi \frac{\partial^2}{\partial x \partial z} \right) \left( \sigma_{zz} \right)
\]

\[
+ \left( \eta_{33} \left( -\nabla^2 \right)^{\gamma_{33}} + \tau_{33} \frac{\partial}{\partial t} \left( -\nabla^2 \right)^{\gamma_{33}/2 - 0.5} \right) \left( \sin^2 \phi \frac{\partial^2}{\partial x^2} + \cos^2 \phi \frac{\partial^2}{\partial z^2} + \sin 2 \phi \frac{\partial^2}{\partial x \partial z} \right) \left( \sigma_{zz} + f_x \right),
\]

\[
\frac{\partial^2 \sigma_{zz}}{\partial t^2} = \left( \eta_{11} \left( -\nabla^2 \right)^{\gamma_{11}} + \tau_{11} \frac{\partial}{\partial t} \left( -\nabla^2 \right)^{\gamma_{11}/2 - 0.5} \right) \left( \cos^2 \phi \frac{\partial^2}{\partial x^2} + \sin^2 \phi \frac{\partial^2}{\partial z^2} - \sin 2 \theta \frac{\partial^2}{\partial x \partial z} \right) \left( \sigma_{xx} \right)
\]

\[
+ \left( \eta_{13} \left( -\nabla^2 \right)^{\gamma_{13}} + \tau_{13} \frac{\partial}{\partial t} \left( -\nabla^2 \right)^{\gamma_{13}/2 - 0.5} \right) \left( \sin^2 \phi \frac{\partial^2}{\partial x^2} + \cos^2 \phi \frac{\partial^2}{\partial z^2} + \sin 2 \phi \frac{\partial^2}{\partial x \partial z} \right) \left( \sigma_{xx} \right)
\]

\[
+ \left( \eta_{33} \left( -\nabla^2 \right)^{\gamma_{33}} + \tau_{33} \frac{\partial}{\partial t} \left( -\nabla^2 \right)^{\gamma_{33}/2 - 0.5} \right) \left( \sin^2 \phi \frac{\partial^2}{\partial x^2} + \cos^2 \phi \frac{\partial^2}{\partial z^2} + \sin 2 \phi \frac{\partial^2}{\partial x \partial z} \right) \left( \sigma_{xx} + f_z \right),
\]

(B-1)

(B-2)
where $\sigma_{xx}$ and $\sigma_{zz}$ denote the horizontal component and vertical component of stress, respectively, $f_s$ and $f_v$ represent the horizontal component and vertical component of source function, respectively. According to the HFDLRD method illustrated previously, Eqs. (B-1) and (B-2) can be reformulated as

$$\begin{align*}
\frac{\partial^2 \sigma_{xx}}{\partial t^2} &= \left( \nu_{11} \tau^{-1} \left( k^2 \nu_{11} J_0^2 \left( \tau \sigma_{xx} \right) \right) + \tau_{11} \tau^{-1} \left( k^2 \nu_{11} J_0^2 \left( \tau \sigma_{xx} \right) \right) \right) \left( \cos^2 \frac{\partial^2}{\partial x^2} - \sin^2 \frac{\partial^2}{\partial z^2} - \sin 2 \phi \frac{\partial^2}{\partial x \partial z} \right) \\
&\quad + \left( \nu_{13} \tau^{-1} \left( k^2 \nu_{13} J_0^2 \left( \tau \sigma_{zz} \right) \right) + \tau_{13} \tau^{-1} \left( k^2 \nu_{13} J_0^2 \left( \tau \sigma_{zz} \right) \right) \right) \left( \cos^2 \frac{\partial^2}{\partial z^2} - \sin^2 \frac{\partial^2}{\partial x^2} - \sin 2 \phi \frac{\partial^2}{\partial y \partial z} \right) + f_s, \\
\frac{\partial^2 \sigma_{zz}}{\partial t^2} &= \left( \nu_{13} \tau^{-1} \left( k^2 \nu_{13} J_0^2 \left( \tau \sigma_{xx} \right) \right) + \tau_{13} \tau^{-1} \left( k^2 \nu_{13} J_0^2 \left( \tau \sigma_{xx} \right) \right) \right) \left( \cos^2 \frac{\partial^2}{\partial z^2} - \sin^2 \frac{\partial^2}{\partial x^2} - \sin 2 \phi \frac{\partial^2}{\partial y \partial z} \right) \\
&\quad + \left( \nu_{13} \tau^{-1} \left( k^2 \nu_{13} J_0^2 \left( \tau \sigma_{zz} \right) \right) + \tau_{13} \tau^{-1} \left( k^2 \nu_{13} J_0^2 \left( \tau \sigma_{zz} \right) \right) \right) \left( \cos^2 \frac{\partial^2}{\partial x^2} - \sin^2 \frac{\partial^2}{\partial z^2} - \sin 2 \phi \frac{\partial^2}{\partial x \partial z} \right) + f_v.
\end{align*}$$

(B-3)

(B-4)

Finally, based on Eq. (31) and the finite-difference method, Eqs. (B-3) and (B-4) can be conveniently solved by the HFDLRD method.

References


