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Two robust weighted Runge-Kutta discontinuous Galerkin methods for scalar wavefield modeling

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ABSTRACT

Developing fast, efficient forward modeling methods with low numerical dispersion is an important research topic in computational geophysics. The discontinuous Galerkin (DG) method shows advantages in suppressing numerical dispersion and supporting flexible meshing, but it faces the disadvantage of slow computation. To accelerate the simulation, one strategy is to relax the CFL condition, thereby allowing larger time steps. Based on the weighted Runge-Kutta discontinuous Galerkin (WRK-DG) method, in which WRK is the time discretization scheme and DG the spatial discretization scheme, we propose two robust WRK time discretization schemes: the coefficient-optimized WRK scheme and the stepwise computation-reduced WRK scheme. These schemes are incorporated into the original DG spatial framework for the efficient solution of the scalar wave equation. In addition, our proposed methods include both the numerical flux-based DG method and the penalty function-based DG method. We establish the numerical stability conditions of these schemes and analyze the numerical dispersion characteristics for P^1 – P^5 elements on rectangular meshes. The results show that the two new methods increase the maximum CFL condition numbers and can effectively suppress numerical dispersion. For the numerical flux-based DG method, compared with the third-order RKDG method, the CFL condition number of the coefficient-optimized WRK-DG method is more than three times larger, and the CFL number of the stepwise computation-reduced WRK-DG method is 1.68–2.1 times larger. For the penalty-based DG method, the coefficient-optimized WRK-DG scheme yields a CFL condition number approximately 2.5 times larger than the third-order RKDG method, while the stepwise computation-reduced WRK-DG scheme achieves CFL numbers about 2.16 times larger. Meanwhile, our numerical experiments demonstrate that the proposed DG methods can greatly improve the computational efficiency. © 2026 The Authors. Publishing services by Elsevier B.V. on behalf of KeAi Communications Co. Ltd. This is an open access article under the CC BY-NC-ND license (<http://creativecommons.org/licenses/by-nc-nd/4.0/>).

1. Introduction

Full waveform inversion (FWI) is an important technique for imaging underground structures with broad applications in resource exploration, medical imaging, and deep Earth structure imaging which is developing rapidly (Tarantola, 1984; Tromp et al., 2005; Tape et al., 2007; Lee et al., 2011; Guasch et al., 2020; Zhao et al., 2021; Dong et al., 2023, 2024; Operto et al., 2023; Aghazade et al., 2024; Jiang et al., 2024; Li and Yao, 2025; Yang et al., 2025). Since the inversion process requires repeated solutions of the wave

equation, the accuracy of forward modeling directly impacts the final imaging results. Common methods for seismic forward modeling include the finite difference (FD) method (Fei and Larnier, 1995; Liu, 2022), the finite element (FE) method (Marfurt, 1984; Chen et al., 2022), the pseudo-spectral method (Kosloff and Baysal, 1982), the spectral element method (Komatitsch et al., 2000; Li et al., 2024), and the discontinuous Galerkin (DG) method (Käser and Dumbser, 2006).

The conventional FD methods have been applied to obtain the solution of wave equations widely (Zhang and Zhang, 2022; Fang et al., 2023; Peng et al., 2023; Daszuta, 2024; Xu and Liu, 2024; Lou et al., 2025; Miao and Zhang, 2025; Wang et al., 2025). However, many FD methods suffer from severe numerical dispersion (Fei and Larnier, 1995). The FE method is another widely used

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forward modeling technique for solving wave equations (Marfurt, 1984; Chen et al., 2022). Compared with FD methods, its key advantage lies in flexible mesh generation. However, the conventional FE method requires solving large-scale linear systems, which results in low efficiency for wavefield simulations.

The DG method was proposed by Reed and Hill (1973). The discontinuous Galerkin (DG) method is an important development derived from the traditional continuous Galerkin finite element method. Key advantages include flexible mesh generation, the ability to accommodate discontinuities between elements, and strong suppression of numerical dispersion (He et al., 2014). Compared with the pseudo-spectral method and the spectral element method, the DG method allows discontinuities between elements and thus can naturally handle discontinuous solutions. There are many formulations of DG, including the numerical flux-based DG method (Cockburn and Shu, 1989) and the interior penalty-based DG method (Wheeler, 1978). For the numerical flux-based DG, Cockburn and Shu (1989) proposed the Runge-Kutta discontinuous Galerkin method, and the numerical dispersion properties of flux-based DG have been analyzed in several studies (Hu et al., 1999; He et al., 2014; Huang et al., 2022). The interior penalty function-based DG has also been successfully applied to solve wave equations (Grote et al., 2006). Numerical dispersion and stability analyses have been carried out by De Basabe et al. (2008) and De Basabe and Sen (2010). In computational geophysics, DG also has wide applications and is developing rapidly (Duru et al., 2021, 2022; Zhang et al., 2021, 2022, 2023; Gulizzi and Saye, 2022; Wolf et al., 2022; Adjerd et al., 2023; Bonetti et al., 2023; Huang et al., 2023, 2024, 2025; Cao and Zhang, 2024).

However, DG also faces the disadvantages of large computational and storage costs (He et al., 2014), so it is necessary to improve the computational efficiency of the DG method to make it suitable for large-scale wavefield simulations. One approach is to use a DG-FD hybrid algorithm, which can combine the computational efficiency of the FD method with the flexible mesh generation of the DG method (Granath and Wang, 2025). In addition, relaxing the Courant-Friedrichs-Lewy (CFL) condition (Courant et al., 1928) of the DG method enables larger time steps, thereby reducing computational time (He et al., 2015, 2023). In this paper, we focus on enhancing the numerical stability of the DG method. In this regard, He et al. (2015) developed a weighted Runge-Kutta DG method. Its core strategy is to render the implicit operator explicit via iterative procedures and introduce a weighting term to further strengthen stability criteria. Qiu et al. (2020) later incorporated an additional weighting factor to further enhance the stability properties of the scheme. Building on previous work, we propose three weighting factors and compute the nearly optimal CFL condition numbers. This method improves stability conditions, thus enabling larger time steps. In addition, He et al. (2015) and Qiu et al. (2020) only considered the application of WRK and double WRK methods to the flux-based DG method and calculated the approximately optimal CFL number. In this work, we not only compute the approximate maximum Courant number for the robust WRK flux-based DG methods, but also extend this analysis to the interior penalty DG method for acoustic wavefield modeling.

This study focuses on the acoustic wave equation. Section 2 introduces the fundamentals of the DG method, including the flux-based DG and the interior penalty DG. Next, we introduce the two robust methods proposed in this work. We then analyze the numerical characteristics of the robust weighted RKDG methods, including numerical stability and numerical dispersion. Finally, a series of numerical experiments are conducted to demonstrate the performance and advantages of our proposed methods.

2. The DG methodology

2.1. Numerical flux-based DG method

The two-dimension acoustic wave equation in a homogeneous medium can be written as follows.

$$\frac{\partial^2 u}{\partial t^2} - c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} \right) = f(t), \quad (1)$$

where c is the sound velocity, u is the displacement, and $f(t)$ is the source function. In this section, we describe the spatial semi-discrete scheme of the DG method based on numerical flux (Cockburn and Shu, 1989; Hu et al., 1999).

Eq. (1) can be transformed into a hyperbolic form by introducing variables p_1 and p_2 satisfying

$$\begin{cases} \frac{\partial p_1}{\partial t} = \frac{\partial u}{\partial x}, \\ \frac{\partial p_2}{\partial t} = \frac{\partial u}{\partial z}. \end{cases} \quad (2)$$

Setting $V = (u, p_1, p_2)^T$, $g = (f, 0, 0)^T$, we further get the following scheme:

$$\frac{\partial V}{\partial t} + \nabla \cdot \theta(V) = g, \quad (3)$$

$$\theta(V) = \begin{pmatrix} -c^2 p_1 & -c^2 p_2 \\ -u & 0 \\ 0 & -u \end{pmatrix}. \quad (4)$$

The regions are first divided into rectangles or triangles in the two-dimensional case $\Omega = \cup_l \Omega_l$. The test function space is

$$V_h = \{q \in L^2(\Omega) : q|_E \in P^k(E), \forall E \in \Omega_l\}, \quad (5)$$

where $P^k(\Omega_l)$ denotes polynomials of degree at most k . Let q be the test function, and multiply it by both sides of the equation.

$$\int_{\Omega_l} q \left(\frac{\partial V}{\partial t} + \nabla \cdot \theta(V) \right) dv = \int_{\Omega_l} q g dv. \quad (6)$$

The following equation can be obtained from Green's formula.

$$\int_{\Omega_l} \left(q \frac{\partial V}{\partial t} - \theta(V) \cdot \nabla q \right) dv + \int_{\partial \Omega_l} (q \theta(V) \cdot \mathbf{n}) ds = \int_{\Omega_l} q g dv. \quad (7)$$

For the boundary term $\partial \Omega_l$, the local Lax-Friedrichs fluxes are selected in this paper (Cockburn and Shu, 1989). The formulation of the local Lax-Friedrichs fluxes can be written as follows:

$$\tilde{F}(V^{\text{int}}, V^{\text{ext}}, n) = \frac{1}{2} \left[\theta(V^{\text{int}}) + \theta(V^{\text{ext}}) \right] \cdot \mathbf{n} - \frac{C_{\partial \Omega_l}}{2} (V^{\text{ext}} - V^{\text{int}}), \quad (8)$$

where $C_{\partial \Omega_l}$ is taken to be the maximum absolute eigenvalue of $\frac{\partial}{\partial V} \theta(V^{\text{int}}) \cdot \mathbf{n}$ and $\frac{\partial}{\partial V} \theta(V^{\text{ext}}) \cdot \mathbf{n}$.

Then we can use basis functions for expansion. Let $\{\phi_i\}_{i=1}^{d_{\text{loc}}}$ be a set of basis functions defined on element Ω_l , and d_{loc} represents the number of degrees of freedom associated with the basis functions. For example, in 2D simulations with rectangular or triangular elements, $d_{\text{loc}} = \frac{(k+1)(k+2)}{2}$. We use Legendre basis functions for rectangular elements and tensor-based basis

functions for triangular meshes. The expansion with those basis functions can be written as Eq. (9).

$$V = \sum_{i=1}^{d_{loc}} V_i^l(t) \phi_i. \quad (9)$$

Replace v with ϕ_i , the semi-discrete form of the flux-based DG method can be obtained as follows.

$$\begin{aligned} & \sum_{i=1}^{d_{loc}} \frac{\partial V_i^l(t)}{\partial t} \int_{\Omega_i} \phi_i \phi_i \, dv - \int_{\Omega_i} \theta \left(\sum_{i=1}^{d_{loc}} V_i^l(t) \phi_i \right) \cdot \left(\frac{\partial \phi_i}{\partial x}, \frac{\partial \phi_i}{\partial z} \right) dv \\ & + \sum_{i=1} \int_{\Omega_i \cap \Omega_{i-}} \phi_i \tilde{F} \left(\left(\sum_{i=1}^{d_{loc}} V_i^l(t) \phi_i \right), \left(\sum_{i=1}^{d_{loc}} V_i^l(t) \phi_i^- \right), \mathbf{n} \right) ds \\ & = \int_{\Omega_i} g \phi_i \, dv, \end{aligned} \quad (10)$$

where ϕ_i^- denotes the basis function on element Ω_{i-} and $\tilde{F}$ denotes the numerical flux as shown in Eq. (8). Setting V as the coefficients $\{V_i^l(t)\}_{i=1}^{d_{loc}}$ over all elements $\{\Omega_i\}$, then Eq. (10) can be transformed into a first-order ODE system after inverting the mass matrix.

2.2. Interior penalty-based DG method

The spatial discretization of symmetric interior penalty Galerkin (SIPG) method can be referred to Grote et al. (2006). Here we list the main steps for the scalar wave equation. The weak form can be written as Eq. (11), where the space of the test function q is same as that in Eq. (5).

$$\int_{\Omega} q \cdot \frac{\partial^2 u}{\partial t^2} dv + \int_{\Omega} c^2 \nabla u \cdot \nabla q dv - \sum_{F \in \partial \Omega_i} \int_F c^2 (\mathbf{n} \cdot \nabla u) q ds = \int_{\Omega} g \cdot q dv. \quad (11)$$

Since the value of u on boundary $F \in \partial \Omega_i$ is discontinuous across elements, we denote the interior and exterior values of u on $\partial \Omega_i$ as u^+ and u^- , respectively, and let $\mathbf{n}^+$ be the outward normal vector from boundary F , and $\mathbf{n}^- = -\mathbf{n}^+$. The average $\{u\}$ and the jump $[u]$ of the function can then be written as follows:

$$\{u\} := \frac{1}{2}(u^+ + u^-), \quad (12)$$

$$[u] := u^+ \mathbf{n}^+ + u^- \mathbf{n}^-. \quad (13)$$

Then the boundary term can be transformed into the following equation:

$$\begin{aligned} J_F(u, v; r) &= - \sum_{F \in \partial \Omega_i} \int_F c^2 (\mathbf{n} \cdot \nabla u) q ds \\ &= - \sum_{F \in \partial \Omega_i} \int_F [q] \cdot \{c^2 \nabla u\} ds + \eta \sum_{F \in \partial \Omega_i} \int_F [u] \cdot \{c^2 \nabla q\} ds \\ &\quad + r \sum_{F \in \partial \Omega_i} \int_F \{c^2\} \cdot [u][q] ds. \end{aligned} \quad (14)$$

Among them, r is called the interior penalty factor. We use the following formulation (Ainsworth et al., 2006): $r = \frac{(k+1)(k+2)}{2e}$, where e represents the length of the element Ω_i . For the SIPG method, the parameter η takes the value -1 (Grote et al., 2006). By

expanding with the basis function $u = \sum_{i=1}^{d_{loc}} u_i(t) \phi_i$, the following equation can be obtained:

$$\begin{aligned} A_{ij}^{(1)} \frac{\partial^2 u_j}{\partial t^2} + A_{ij}^{(2)} u_j + A_{ij}^{(3)} u_j &= F_i, \\ A_{ij}^{(1)} &= \sum_{E \in \Omega_i} \int_E \phi_j \cdot \phi_i \, dv, \\ A_{ij}^{(2)} &= \sum_{E \in \Omega_i} \int_E \nabla \phi_j \cdot \nabla \phi_i \, dv, \\ A_{ij}^{(3)} &= \sum_{F \in \partial \Omega_i} J_F(\phi_j, \phi_i; r), \\ F_i &= \sum_{E \in \Omega_i} \int_E f \cdot \phi_i \, dv, \\ i, j &= 1, \dots, d_{loc}. \end{aligned} \quad (15)$$

Setting U as the coefficients $\{u_i(t)\}$, and further setting $\frac{\partial U}{\partial t} = \hat{U}$ and $W = \begin{pmatrix} U \\ \hat{U} \end{pmatrix}$, the second-order ordinary differential equation can be reformulated as a first-order ODE system.

3. Robust weighted Runge-Kutta methods

For the ordinary differential equation (ODE) given in Eq. (15), we propose two methods: the coefficient-optimized weighted RK scheme and the stepwise computation-reduced weighted RK scheme. These two methods allow the use of a larger time-step size, and we therefore refer to them as robust weighted Runge-Kutta discontinuous Galerkin (WRKDG) methods. The two schemes are introduced separately in the following sections.

3.1. Coefficient-optimized weighted RK scheme

After space discretization, Eqs. (10) and (15) can be transformed into an ordinary differential equation. While the variable definitions of flux-based DG and IPDG differ, the computational processes for variables W and V remain the same, so we will use W as an example in this section.

$$\frac{\partial W(t)}{\partial t} = L(W(t)) + F. \quad (16)$$

Among the above equations, the specific expression of L is provided by Eqs. (10) and (15) after inverting the mass matrix. It should be noted that the explicit form of L differs between the flux-based DG and IPDG methods. However, for notational convenience, it is uniformly denoted as the matrix L throughout this work. For ordinary differential equations, the classic time discretization scheme is the TVD RK time discretization scheme (Cockburn and Shu, 1989). The third-order explicit Runge-Kutta method can be described as Eq. (17).

$$\begin{cases} W^{(0)} = W^{(n)}, \\ W^{(1)} = W^{(0)} + \Delta t L(W^{(0)}) + \Delta t F(t^n), \\ W^{(2)} = \frac{3}{4} W^{(0)} + \frac{1}{4} W^{(1)} + \frac{1}{4} \Delta t L(W^{(1)}) + \frac{1}{4} \Delta t F(t^n + \Delta t), \\ W^{(n+1)} = \frac{1}{3} W^{(0)} + \frac{2}{3} W^{(2)} + \frac{2}{3} \Delta t L(W^{(2)}) + \frac{2}{3} \Delta t F\left(t^n + \frac{\Delta t}{2}\right). \end{cases} \quad (17)$$

We consider the following third-order diagonally implicit Runge-Kutta scheme (Hairer et al., 1993):

$$W^{(n+1)} = W^{(n)} + \frac{\Delta t}{2} (K^{(n)} + \bar{K}^{(n)}). \quad (18)$$

$$\begin{cases} K^{(n)} = L(W^{(n)} + r\Delta t K^{(n)}) + F(t^n + r\Delta t), \\ \bar{K}^{(n)} = L(W^{(n)} + (1-2r)\Delta t K^{(n)} + r\Delta t \bar{K}^{(n)}) + F(t^n + (1-r)\Delta t), \end{cases} \quad (19)$$

where $r = \frac{1}{2} - \frac{\sqrt{3}}{6}$.

He et al. (2015) proposed a weighted Runge-Kutta DG method by transforming an implicit scheme into an explicit one. Based on their work, we modified the method. Specifically, we modify the weighting strategy and increase the number of weights in the scheme, which can further improve stability. Ignoring the source term, the scheme for the new method is described below.

$$K^{(n)} = r\Delta t L(K^{(n)}) + L(W^{(n)}).$$

If $\|r\Delta t\| < 1$, then the iteration process can be conducted as follows.

$$\begin{cases} K_0^{(n)} = L(W^{(n)}), \\ K_1^{(n)} = r\Delta t L(K_0^{(n)}) + L(W^{(n)}), \\ K_2^{(n)} = r\Delta t L(K_1^{(n)}) + L(W^{(n)}). \end{cases} \quad (20)$$

To further enhance the stability of the method, we introduce two weighting factors η_1 and η_2 , and the scheme can be written as follows:

$$\begin{aligned} K^{(n)} &= \eta_1 K_2^{(n)} + \eta_2 K_1^{(n)} + (1 - \eta_1 - \eta_2) K_0^{(n)} \\ &= L(W^{(n)}) + (\eta_1 + \eta_2) r\Delta t L^2(W^{(n)}) \\ &\quad + \eta_2 (r\Delta t)^2 L^3(W^{(n)}). \end{aligned} \quad (21)$$

We can set $T^{(n)} = W^{(n)} + (1 - 2r)K^{(n)}$. Then we have

$$\bar{K}^{(n)} = r\Delta t L(\bar{K}^{(n)}) + L(T^{(n)}). \quad (22)$$

Following the iterative procedure described above, we obtain Eq. (23).

$$\begin{cases} \bar{K}_0^{(n)} = L(T^{(n)}), \\ \bar{K}_1^{(n)} = r\Delta t L(\bar{K}_0^{(n)}) + L(T^{(n)}), \\ \bar{K}_2^{(n)} = r\Delta t L(\bar{K}_1^{(n)}) + L(T^{(n)}). \end{cases} \quad (23)$$

Then we can derive

$$\begin{aligned} \bar{K}^{(n)} &= \eta_1 \bar{K}_2^{(n)} + \eta_2 \bar{K}_1^{(n)} + (1 - \eta_1 - \eta_2) \bar{K}_0^{(n)} \\ &= L(T^{(n)}) + (\eta_1 + \eta_2) r\Delta t L^2(T^{(n)}) + \eta_2 (r\Delta t)^2 L^3(T^{(n)}). \end{aligned} \quad (24)$$

In addition, we note that the works by He et al. (2015) and Qiu et al. (2020) are both based on $r = \frac{1}{2} - \frac{\sqrt{3}}{6}$. We can further optimize the value of r by incorporating the two previous weights, resulting

in a total of three adjustable weights. The specific selection of these parameters will be discussed in the following section.

3.2. Stepwise computation-reduced weighted RK scheme

The weighted Runge-Kutta methods require multiple evaluations of the function $L(W)$ in Eq. (16). Since this process involves calculating the spatial operator, the computational cost of each step is relatively high. Now we aim to reduce the computational cost each time step and increase the Courant number as much as possible. The stepwise computation-reduced weighted RK is similar to the coefficient-optimized weighted RK scheme but differs in its computational cost per time step and weighting strategy.

Since $W^{(n+1)} = W^{(n)} + \frac{\Delta t}{2} (K^{(n)} + \bar{K}^{(n)})$, we also need to calculate $K^{(n)}$ and $\bar{K}^{(n)}$. Then $K^{(n)}$ can be written as the weighted form of the above expression (20), where η_1 is the new weight.

$$K^{(n)} = \eta_1 K_2^{(n)} + (1 - \eta_1) K_0^{(n)}. \quad (25)$$

In the same way, we can make the expression of $\bar{K}^{(n)}$ explicit. One iteration then yields the following equation.

$$\begin{cases} \bar{K}_0^{(n)} = L(T^{(n)}), \\ \bar{K}_1^{(n)} = r\Delta t L(\bar{K}_0^{(n)}) + L(T^{(n)}). \end{cases} \quad (26)$$

Unlike the coefficient-optimized WRK scheme, the calculation of $\bar{K}^{(n)}$ is performed only once, thereby reducing the computational cost. Then $\bar{K}^{(n)}$ can be expressed as a weighted combination of the two formulas above, where η_2 is the weighting factor.

$$\bar{K}^{(n)} = \eta_2 \bar{K}_1^{(n)} + (1 - \eta_2) \bar{K}_0^{(n)}. \quad (27)$$

Similar to the coefficient-optimized WRK scheme, we also take r as a parameter. In the next section, we consider the stability conditions as the parameters η_1 , η_2 , and r vary. Through numerical stability calculations, we find that the method exhibits excellent numerical stability when η_1 , η_2 , and r lie between 0 and 1. Therefore, we restrict their values to this interval.

4. Numerical properties

4.1. Stability conditions

For the robust WRKDG methods, their CFL condition numbers can be derived. The detailed process is listed in appendix A. Based on these results, we adjusted the weight parameters incorporated in our proposed methods to achieve approximately optimal Courant numbers, as shown in Tables 1 and 2, respectively. Since the spatial discretization schemes differ between the flux-based DG and IPDG methods, the selection of the optimal parameters corresponding to the approximately optimal Courant number also

Table 1
Maximum courant number of different methods for the flux DG schemes.

	p^1	p^2	p^3	p^4	p^5
Coefficients-optimized	0.99840	0.57042	0.36471	0.26110	0.19763
WRKDG					
Classic 3rd RK DG	0.31410	0.16475	0.11039	0.07510	0.05582
Rate	3.17860	3.46233	3.30383	3.47669	3.54048
WRKDG	1.096	0.338	0.240	0.162	0.121
Double WRKDG	1.032	0.381	0.205	-	-
Stepwise computation-reduced WRKDG	0.65374	0.30891	0.19230	0.12665	0.09413
Rate	2.08131	1.87502	1.74200	1.68641	1.68631

Table 2
Maximum courant number of different methods for the IP DG schemes.

	p^1	p^2	p^3	p^4	p^5
Coefficients-optimized WRKDG	0.71980	0.39425	0.25477	0.17414	0.12709
Classic 3rd RK DG	0.28867	0.15811	0.10217	0.06984	0.05097
Rate	2.49350	2.49351	2.49358	2.49341	2.49342
Stepwise computation-reduced WRKDG	0.62460	0.34211	0.22108	0.15111	0.11028
Rate	2.16371	2.16374	2.16384	2.16365	2.16362

Table 3
The parameters corresponding to the approximate maximum courant numbers of different methods.

	p^1	p^2	p^3	p^4	p^5
Coefficients-optimized WRK-flux DG	$\eta_1 = 0.6$ $\eta_2 = 0.4$ $r = 0.19$	$\eta_1 = 0.7$ $\eta_2 = 0.4$ $r = 0.22$	$\eta_1 = 0.7$ $\eta_2 = 0.4$ $r = 0.22$	$\eta_1 = 0.7$ $\eta_2 = 0.4$ $r = 0.22$	$\eta_1 = 0.7$ $\eta_2 = 0.4$ $r = 0.22$
Coefficients-optimized WRK-IPDG	$\eta_1 = 0.5$ $\eta_2 = 0.5$ $r = 0.35$	$\eta_1 = 0.5$ $\eta_2 = 0.5$ $r = 0.35$	$\eta_1 = 0.5$ $\eta_2 = 0.5$ $r = 0.35$	$\eta_1 = 0.5$ $\eta_2 = 0.5$ $r = 0.35$	$\eta_1 = 0.5$ $\eta_2 = 0.5$ $r = 0.35$
Stepwise computation-reduced WRK-flux DG	$\eta_1 = 1$ $\eta_2 = 1$ $r = 0.16$	$\eta_1 = 1$ $\eta_2 = 1$ $r = 0.2$	$\eta_1 = 1$ $\eta_2 = 1$ $r = 0.22$	$\eta_1 = 1$ $\eta_2 = 1$ $r = 0.23$	$\eta_1 = 1$ $\eta_2 = 1$ $r = 0.23$
Stepwise computation-reduced WRK-IPDG	$\eta_1 = 1$ $\eta_2 = 1$ $r = 0.37$	$\eta_1 = 1$ $\eta_2 = 1$ $r = 0.37$	$\eta_1 = 1$ $\eta_2 = 1$ $r = 0.37$	$\eta_1 = 1$ $\eta_2 = 1$ $r = 0.37$	$\eta_1 = 1$ $\eta_2 = 1$ $r = 0.37$

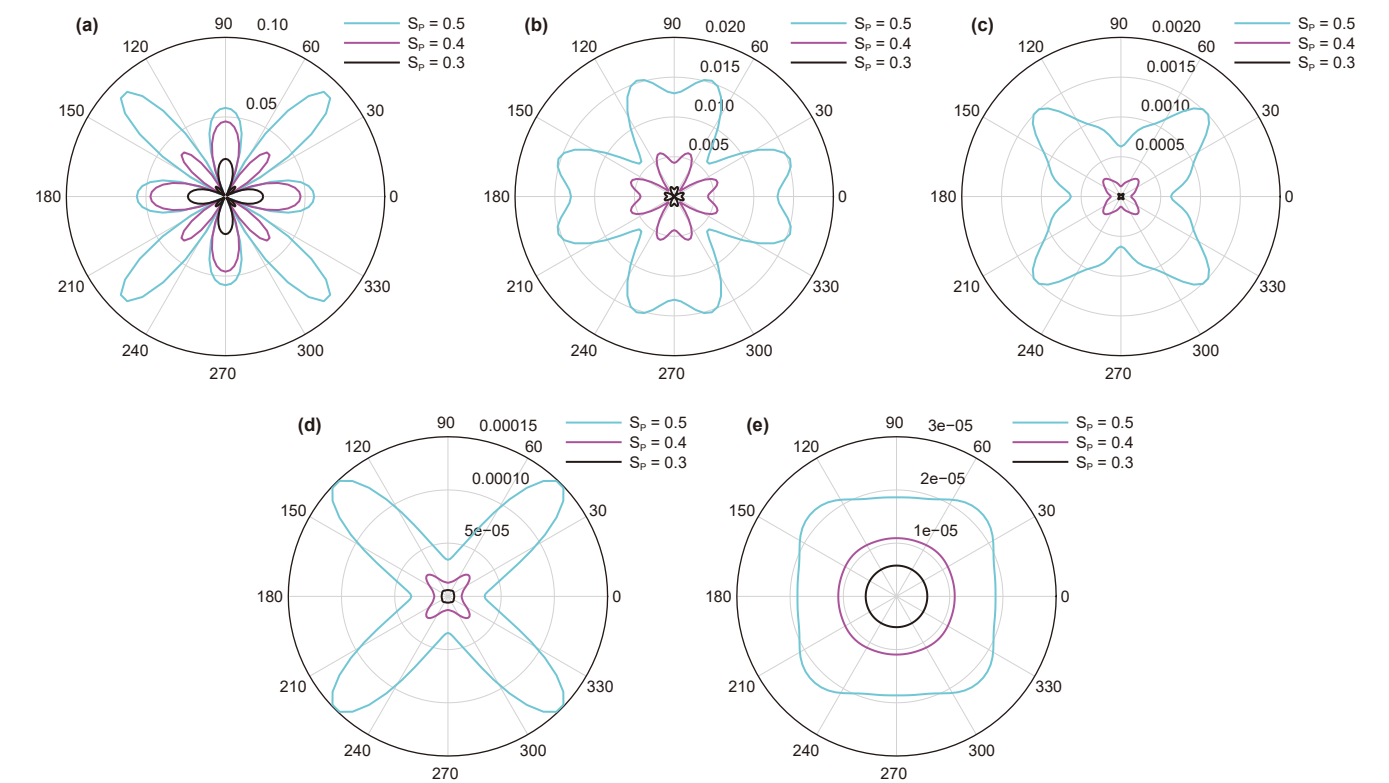


Fig. 1. Numerical dispersion results for the coefficients-optimized WRK-flux DG, (a)–(e) show the P^1 – P^5 elements, respectively.

varies. We list the parameter choices that yield approximately optimal stability in Table 3. For comparison, we also present the numerical stability conditions of the third-order classic RK DG method on rectangular meshes (He et al., 2014, 2019; Qiu et al., 2020).

Table 1 lists the approximate Courant numbers for different methods based on the numerical flux-based DG method. The

second row presents the CFL number conditions of the coefficients-optimized weighted RK scheme using P^1 – P^5 elements. For comparison, the third row provides the maximum Courant number of the third-order RKDG method, while the fourth row shows the ratio of the values in the second row to those in the third row. For the stepwise computation-reduced WRKDG method, the stability conditions are listed in the seventh row, with the

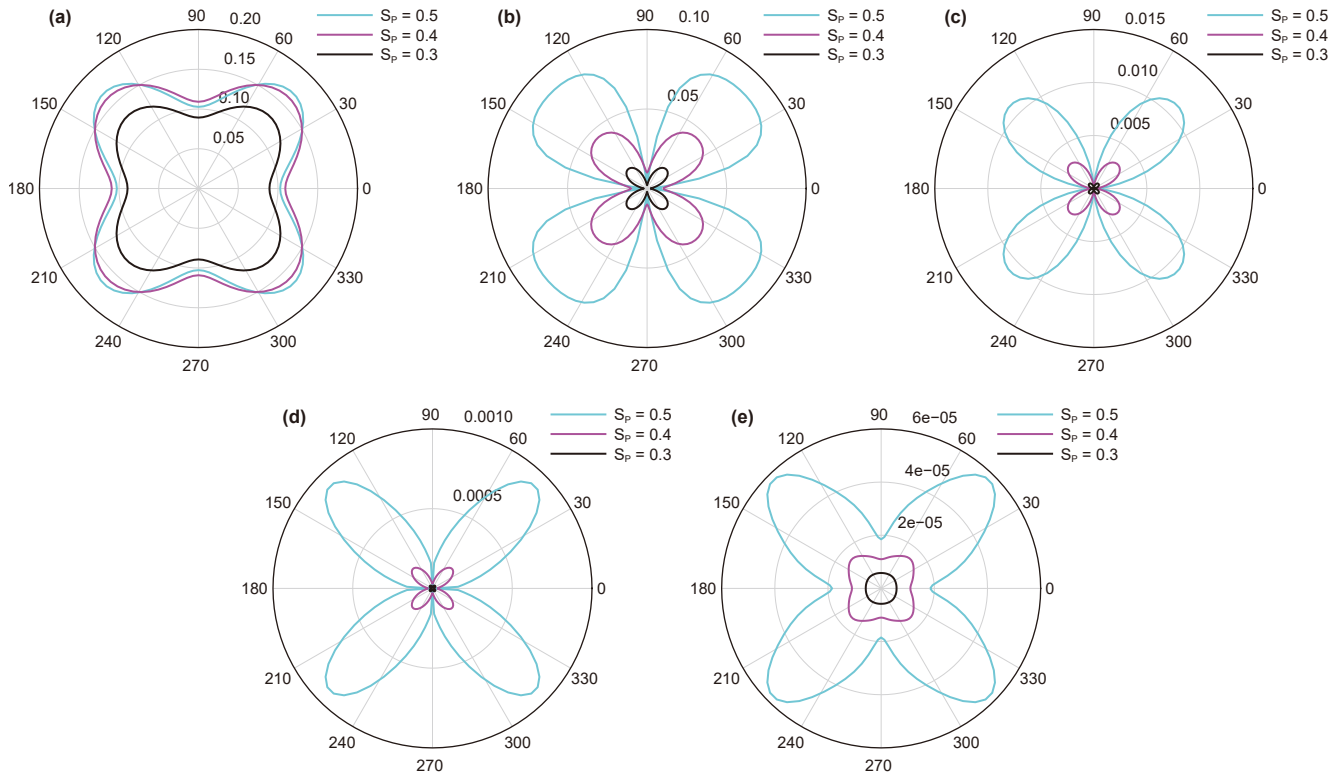


Fig. 2. Numerical dispersion results for the coefficients-optimized WRK - IPDG, (a)–(e) show the P^1 – P^5 elements, respectively.

corresponding ratios to the third-order RKDG method in the last row. The Courant numbers for the interior penalty-based DG methods are presented in Table 2.

For the flux-based DG method with P^1 – P^5 elements, the coefficients-optimized WRKDG method significantly increases the maximum Courant numbers compared with the third-order RKDG method. For P^2 – P^5 elements, which are commonly used in wave-field modeling, this ratio exceeds 3.3, allowing for a substantial increase in the time step. For the stepwise computation-reduced WRKDG scheme, the Courant numbers are approximately 1.68–2.1 times those of the third-order RKDG method. These

results demonstrate that the proposed schemes effectively extend the Courant number, thereby enabling larger time steps. For comparison, the numerical stability of the WRKDG method and the double-weighted WRKDG method is listed in Table 1 (He et al., 2015; He and Yue, 2019; Qiu et al., 2020). It should be noted that Qiu et al. (2020) only applied the double-weighted DG method to P^1 – P^3 elements, and no numerical results are available for the P^4 – P^5 elements of this method. Our methods present two key advantages over the existing approaches: (1) For coefficients-optimized WRKDG, compared with the traditional WRKDG method, our method significantly improves the CFL condition

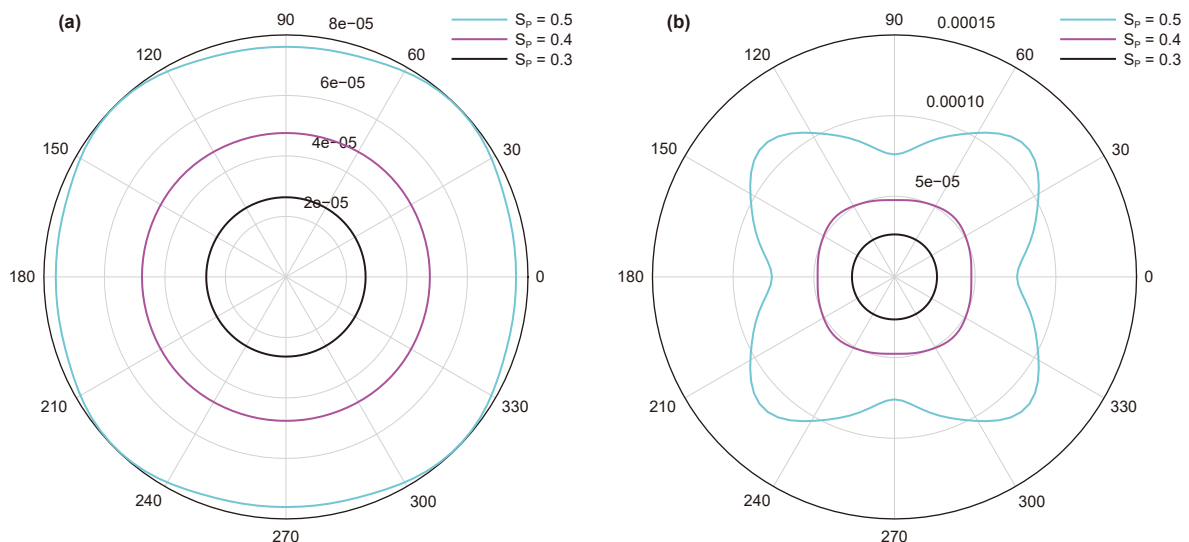


Fig. 3. Numerical dispersion for the stepwise computation-reduced WRKDG for the P^5 element: (a) flux-based DG, (b) interior penalty DG.

number for P^2 – P^5 elements—the most commonly used elements in practical numerical simulations. For the stepwise computation-reduced WRKDG, the computational cost per step is reduced; consequently, the computational efficiency is improved for the P^2 element, as validated by the time comparison results for the Marmousi model in the subsequent sections. (2) The traditional WRKDG method was exclusively applied to flux DG methods, and no relevant applications to IPDG methods have been reported. In contrast, our proposed methods are applicable to both flux-based DG and IPDG methods, thus expanding the application scope of the WRKDG framework.

From Table 2, we observe that for the interior penalty-based DG method, both the coefficients-optimized WRKDG and the stepwise computation-reduced improved WRKDG scheme can effectively extend the maximum CFL number. Specifically, the coefficients-optimized WRKDG scheme increases the CFL numbers by approximately 2.5 times compared with the third-order RKDG

method, while the stepwise computation-reduced WRKDG increases them by about 2.2 times. Thus, our methods enable significantly larger time steps.

From the results of the numerical stability analysis, our two methods exhibit favorable numerical stability and thus possess robust characteristics. Consequently, our robust WRKDG methods can still perform stable computations even when large time steps are adopted, and this finding is further validated by the subsequent numerical experiments.

4.2. Numerical dispersion

Following several works (De Basabe et al., 2008; He et al., 2014, 2019), we summarize the key results from the numerical dispersion analysis of the robust WRKDG methods. The numerical dispersion relations on rectangular meshes are illustrated in Figs. 1 and 2. For the convenience of comparison, the parameters for the

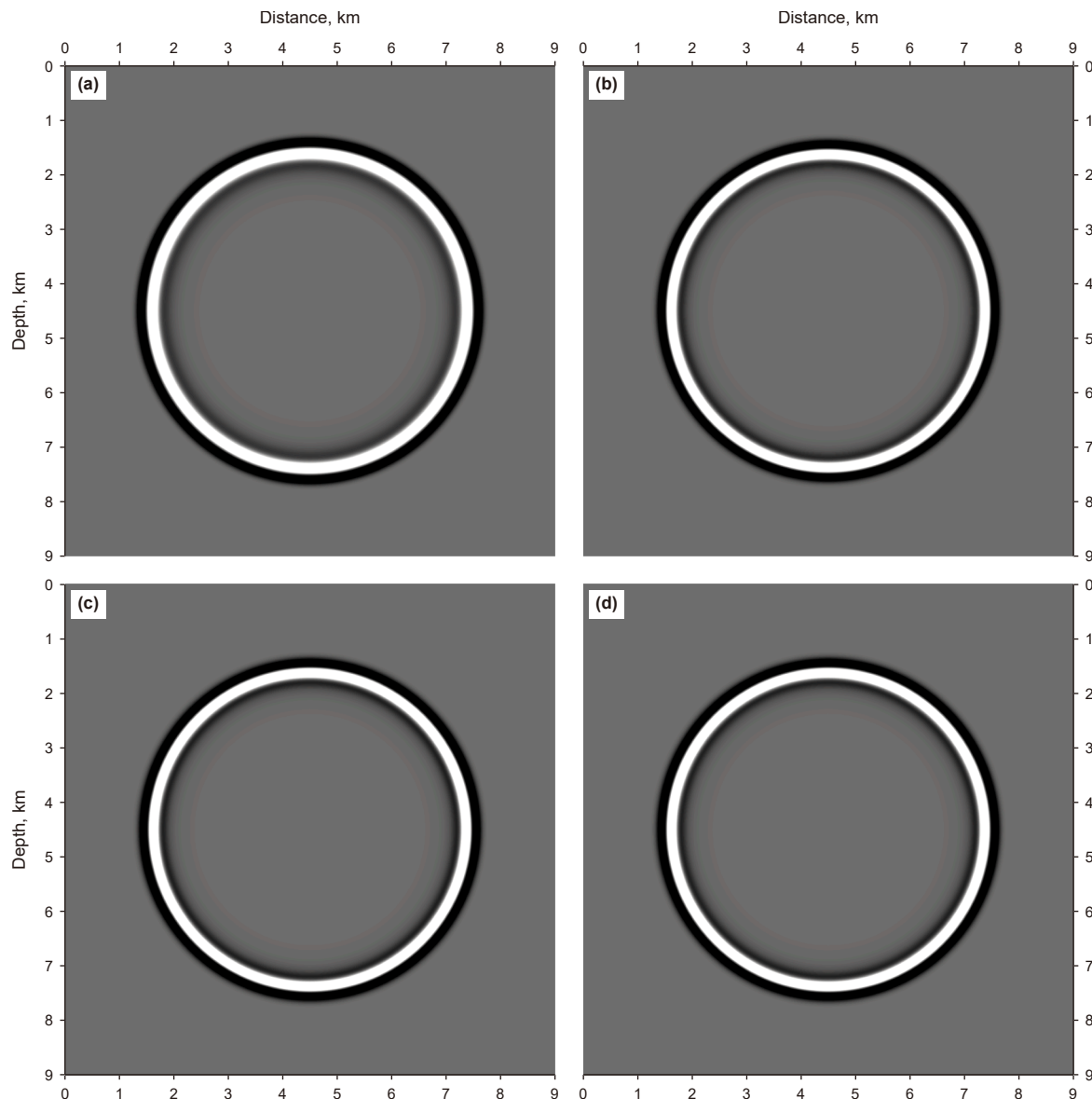


Fig. 4. Wavefield snapshots at 1.0 s: (a) coefficients-optimized WRK-flux DG, (b) coefficients-optimized WRK-IPDG, (c) stepwise computation-reduced WRK-flux DG, (d) stepwise computation-reduced WRK-IPDG.

coefficients-optimized WRKDG scheme are selected as $\eta_1 = 0.5$, $\eta_2 = 0.5$, and $r = 0.35$, while those for the stepwise computation-reduced WRKDG scheme are selected as $\eta_1 = 1$, $\eta_2 = 1$, and $r = 0.37$. These parameter sets yield the approximate maximum Courant numbers achievable with the IPDG. The Courant number is fixed at 0.06 in this section. The numerical dispersion results for the coefficients-optimized WRK for flux DG are presented in Fig. 1, and the coefficients-optimized WRK for IPDG are shown in Fig. 2. Furthermore, the numerical dispersion behaviors of P^1 – P^4 elements in the stepwise computation-reduced WRKDG are similar to those of the coefficients-optimized WRKDG method, so only the numerical dispersion results for the P^5 element are presented in Fig. 3. From the Figures, we can observe that the robust WRKDG methods can effectively suppress numerical dispersion for P^2 – P^5 element.

We analyze the relationship between numerical dispersion and propagation angle. For the IPDG method, both the coefficients-optimized and stepwise computation-reduced WRKDG schemes exhibit similar numerical dispersion characteristics and have significant anisotropy. The maximum numerical dispersion occurs at angles of 45° , 135° , 225° and 315° .

In contrast, for the flux-based DG method, the angle corresponding to the maximum numerical dispersion is not fixed for basis functions of different orders. For the P^1 , P^3 and P^4 elements, the maximum dispersion occurs at 45° . However, for the P^2

element, the dispersion error is relatively small at 45° and becomes larger at approximately 75° . Furthermore, we observe that for the P^5 element, the coefficients-optimized WRKDG scheme differs significantly from the stepwise computation-reduced WRKDG scheme, with the latter exhibiting larger numerical dispersion under the chosen parameters.

5. Numerical experiments

In this section, we perform several numerical experiments to evaluate the dispersion suppression effect and computational efficiency of our method. For the subsequent numerical experiments, the parameters of both our proposed methods are selected in accordance with those listed in Table 3 for implementation.

5.1. Homogeneous model

In this experiment, we first verify the correctness of the proposed method by comparing its results with the analytical solution. Both the numerical flux-based DGM and interior penalty-based DGM are applied in this experiment. The computational domain is set to $0 \text{ km} \leq x, z \leq 9 \text{ km}$, with a spatial step size of 60 m. The wave speed of the medium is 3.28 km/s. A seismic source is located at the center of the computational domain, with a frequency of 15 Hz. In this experiment, we use the zero Dirichlet

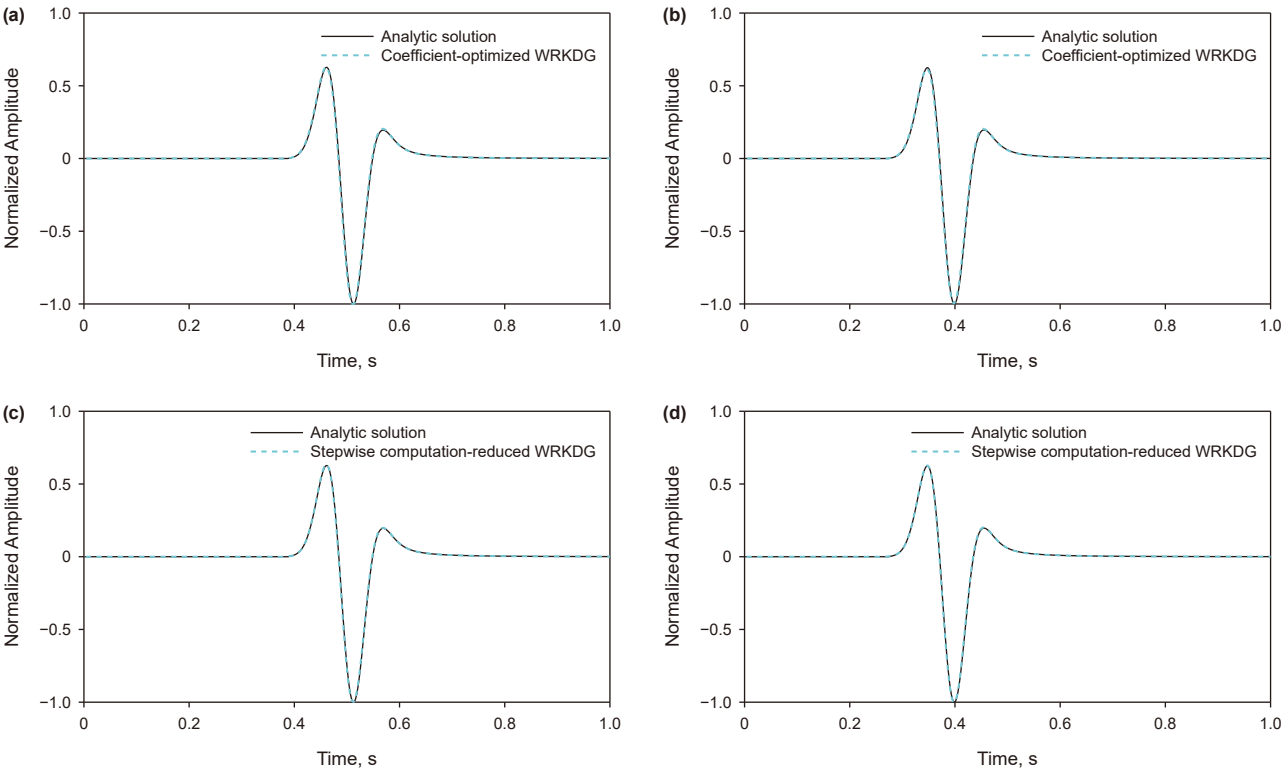


Fig. 5. Waveform comparison for the coefficients-optimized WRK-IPDG with analytic solution at (a) (3.6 km, 5.4 km), (b) (4.5 km, 5.4 km), and the stepwise computation-reduced WRK-flux DG (c) (3.6 km, 5.4 km), (d) (4.5 km, 5.4 km).

Table 4
Time comparison in seconds with different methods for the flux DG at $T = 1.0$ s.

Methods	Classic RK DG	Coefficients-optimized WRKDG	Rate	Stepwise computation-reduced WRKDG	Rate
P^3 element	571.88	341.25	1.68	511.44	1.12
P^4 element	1559.78	829.31	1.88	1261.33	1.23

Table 5
Comparison of computational time in seconds for different IPDG methods at $T = 1.0$ s.

Methods	Classic RK DG	Coefficients-optimized WRKDG	Rate	Stepwise computation-reduced WRKDG	Rate
P^3 element	102.50	76.47	1.34	68.17	1.50
P^4 element	202.48	165.23	1.22	147.44	1.37

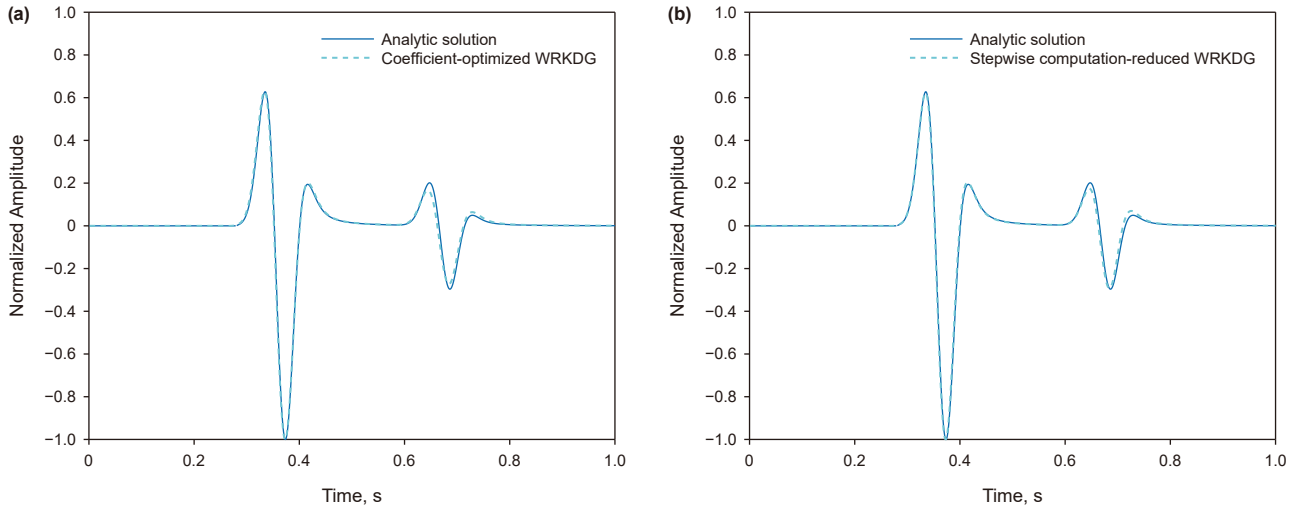


Fig. 6. The receiver for (a) the coefficients-optimized weighted RK DG and (b) the stepwise computation-reduced weighted RK DG.

boundary condition. Numerical experiments are conducted using P^3 and P^4 elements. Wavefield snapshots for the P^3 element are presented in Fig. 4 for the flux-based DG method (a) (b) and the IPDG method (c) (d), with no visible numerical dispersion observed. In addition, Fig. 5 presents comparisons between the numerical results of our methods and the analytical solution derived via the Cagniard-de Hoop method (De Hoop, 1960) at the receiver positions. It can be seen that our improved methods yield results that are in excellent agreement with the analytical solution.

We also compare the computational time of the improved methods with the classic third-order RKDG method for a 1.0 s wavefield simulation, as presented in Tables 4 and 5. The results indicate that our method effectively reduces computation time and improves efficiency compared with the third-order RKDG method. Among the tested schemes, the numerical flux-based coefficients-optimized WRKDG demonstrates the most significant improvement in computational efficiency.

5.2. Two-layer model

In this section, we simulate wave propagation in a velocity model containing strong discontinuities to evaluate the performance of our method. The experiment is similar to that described in He et al. (2019). We apply the flux-based DG method for this experiment. The computational domain is $0 \text{ km} \leq x, z \leq 4 \text{ km}$, the velocity structure consists of two layers (2 km/s and 5 km/s) separated by an interface at $z = 1.6 \text{ km}$. A seismic source is located at (2 km, 2.48 km), the source frequency is 20 Hz, and a receiver is located at (2 km, 1.92 km). Simulations are conducted using P^2 elements. Dirichlet boundary conditions are applied.

Waveforms comparing our numerical results with the analytic solutions (De Hoop, 1960) are presented in Fig. 6. The results show that the improved methods match well with the analytic solution and accurately simulate wavefield propagation across velocity discontinuities.

5.3. Curved boundary model

In this experiment, we simulate a model with a curved boundary. The computational domain is $0 \text{ km} \leq x, z \leq 4 \text{ km}$. Dirichlet boundary conditions are adopted in this experiment. The structure of the model is shown in Fig. 7. A seismic source is located at (2 km, 1.7 km). To more accurately capture the model's structural features, we adopt irregular triangulation, which has a total of 23288 elements. The medium velocities are set to 2.1 km/s (lower region) and 3.1 km/s (upper region), respectively.

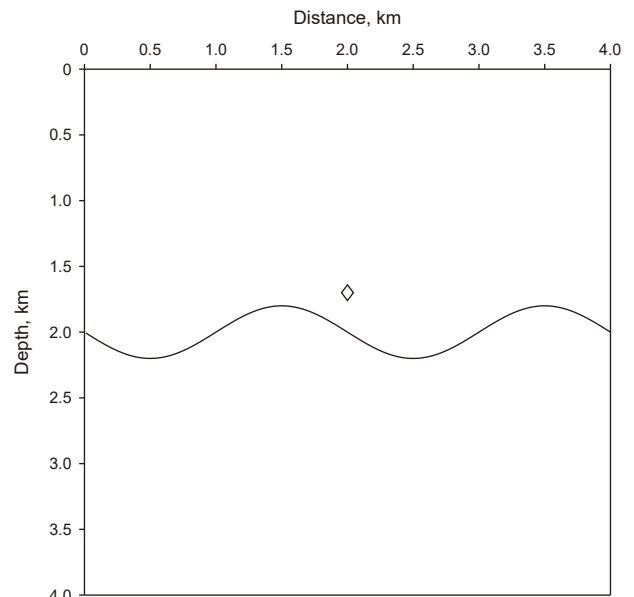


Fig. 7. The structure of the layered model.

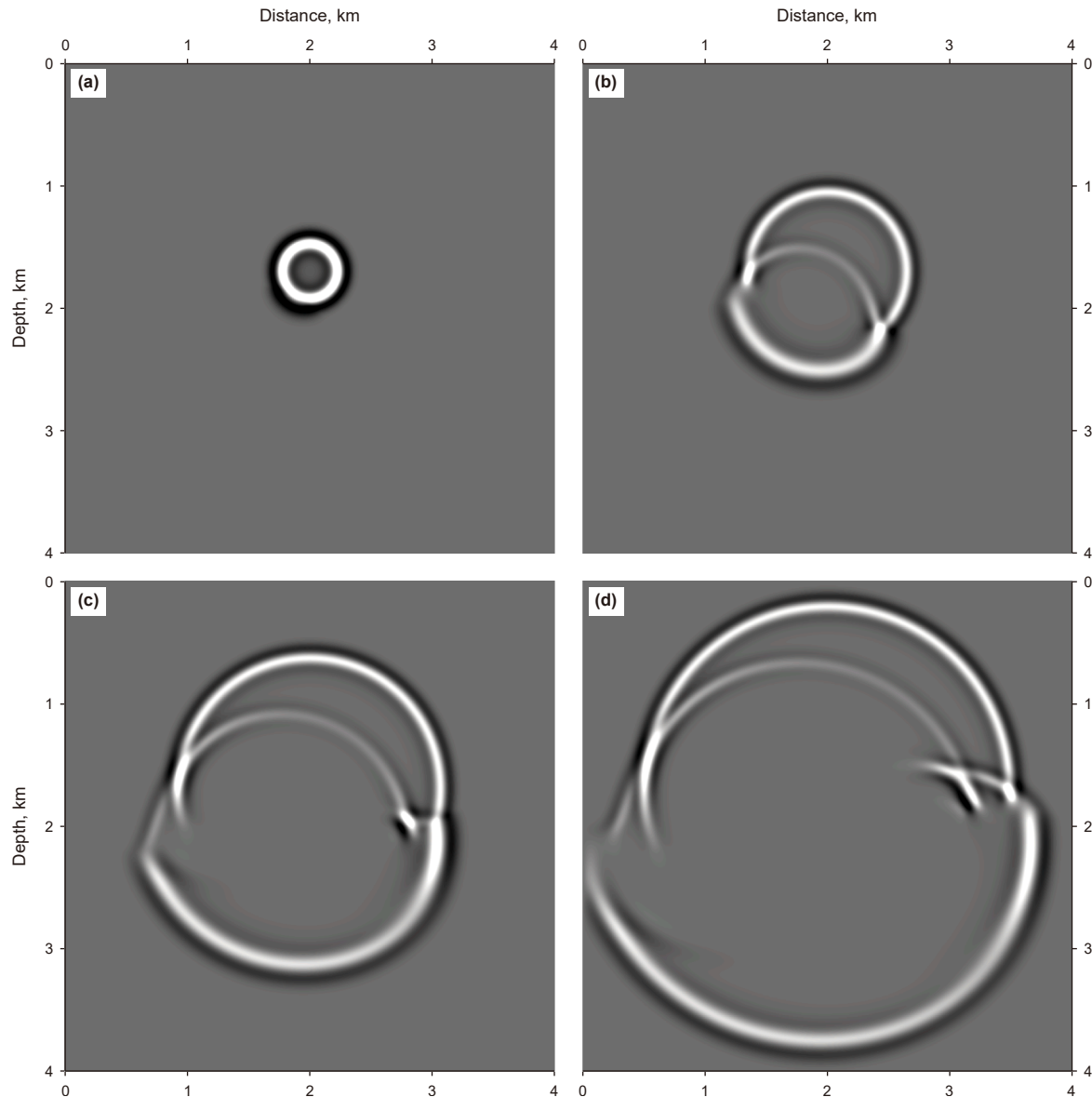


Fig. 8. Wavefield snapshots at different moments: (a) $T = 0.2$ s, (b) $T = 0.4$ s, (c) $T = 0.6$ s, (d) $T = 0.8$ s.

Simulations are performed using the interior penalty-based coefficients-optimized P^3 element, with parameter settings consistent with those used for isotropic media. Wavefield snapshots at $T = 0.2$ s, 0.4 s, 0.6 s and 0.8 s are presented in Fig. 8. These snapshots clearly show the wave propagation process across the curved boundary. Thus, our improved method is shown to effectively simulate wavefields passing through complex curved velocity discontinuities.

5.4. Marmousi model

We perform computations on the Marmousi model (Versteeg and Grau, 1991). The velocity model is shown in Fig. 9. The number of mesh partitions is 414×152 . The P^2 flux-based DG method is employed, using the approximate largest Courant number feasible for the robust WRKDG methods. Sponge boundary conditions are applied to the four boundaries of the computational

domain for wave absorption, with 15 absorbing layers set at each boundary (Cerjan et al., 1985). The seismic source is an 18 Hz Ricker wavelet located at (4.608 km, 0.144 km). Wavefield snapshots at $T = 0.5$ s and $T = 1.0$ s are presented in Fig. 10. Fig. 11 presents the receiver records near the ground surface. It can be observed that the proposed method can be effectively combined with the aforementioned boundary conditions. In future work, we will investigate the integration of this method with PML-type boundary conditions (He et al., 2015). Table 6 summarizes the computation times for different methods at $T = 1.0$ s. Compared with the classic third-order RK DG method, our method improves computational efficiency. We also compare our method with the conventional WRKDG method (He et al., 2015) and the double weighted WRKDG method (Qiu et al., 2020). Results show that both of our improved schemes can further reduce computational time. Furthermore, Table 6 indicates that for the numerical flux-based DG method, the coefficients-optimized WRKDG is more

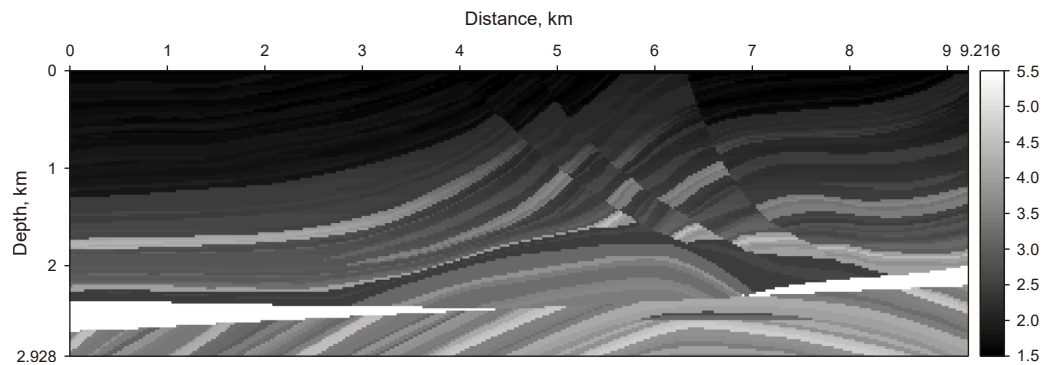


Fig. 9. Acoustic velocity structure of the 2D Marmousi model.

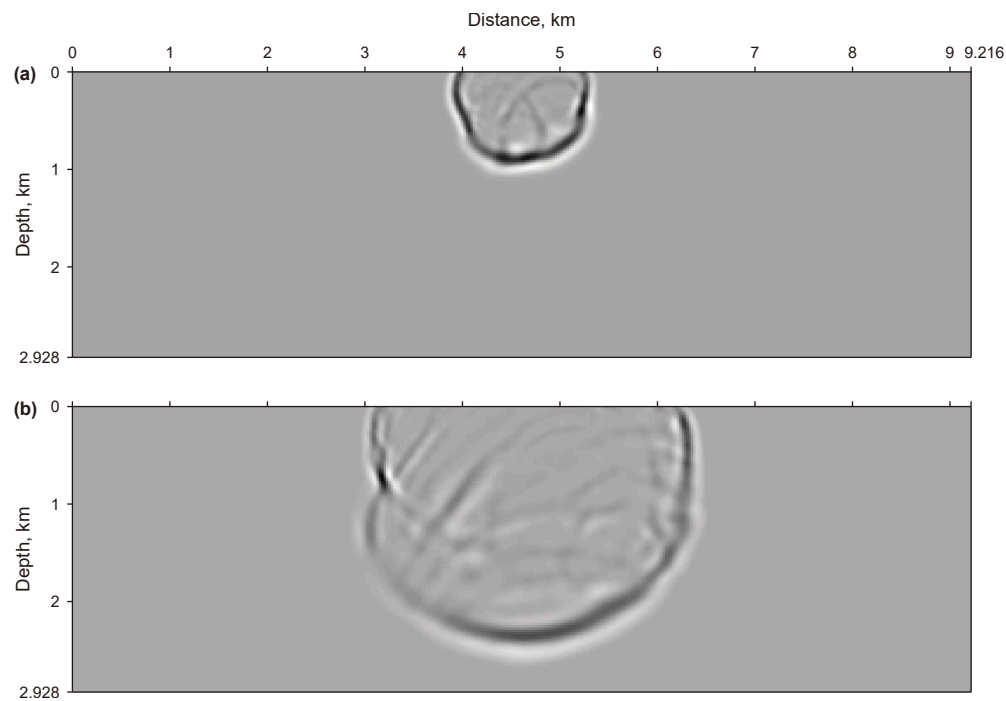


Fig. 10. Wavefield snapshots of the 2D Marmousi model at (a) $T = 0.5$ s and (b) $T = 1.0$ s.

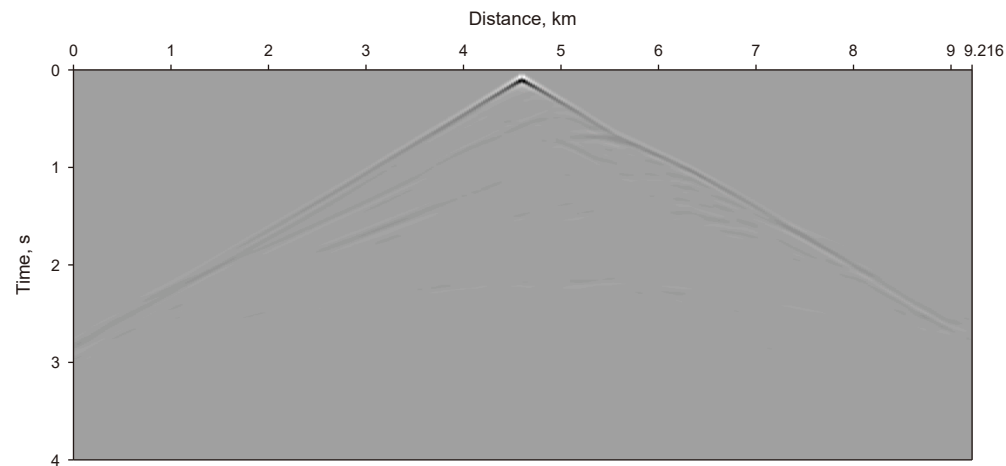


Fig. 11. Near-surface receiver records from 0 to 4 s.

Table 6
Comparison of computational time for different methods.

Methods	Computation time, s
Classic RK DG	2396.55
Conventional WRKDG	2311.10
Double-weighted WRKDG	2097.73
Coefficients-optimized WRKDG	1402.75
Stepwise computation-reduced WRKDG	2030.31

efficient than the stepwise computation-reduced WRKDG, the efficiency of the coefficients-optimized WRKDG is approximately 1.7 times that of the third-order RKDG method.

6. Conclusions

In this paper, we propose two robust weighted Runge-Kutta discontinuous Galerkin methods for solving the acoustic wave equation. We construct two novel WRK schemes that incorporate three variable weighting factors into the WRK coefficients. These robust WRK schemes are combined with both numerical flux-based DG and interior penalty DG methods, and corresponding approximately optimal weighting coefficients are derived for P^1 – P^5 elements. Numerical results demonstrate that the proposed schemes significantly improve computational efficiency. We also conduct a numerical dispersion analysis for the robust WRKDG schemes, which confirms that the proposed method preserves excellent dispersion-suppression performance.

We perform wavefield modeling in both isotropic and two-layer media. The numerical results agree well with the analytical solutions, which validates the effectiveness of the proposed methods. In addition, we perform a numerical experiment on the complex Marmousi velocity model. Tests on rectangular grids demonstrate that the Coefficients-optimized and stepwise computation-reduced DG methods proposed in this work can reduce computational cost and substantially improve efficiency. We further test a curved-boundary model with irregular triangular meshes and demonstrate that the proposed methods effectively suppress numerical dispersion even on unstructured triangular grids.

In summary, we have developed two efficient DG methods for acoustic wavefield modeling. Future work will focus on extending these methods to elastic wave equations.

CRediT authorship contribution statement

Si-Biao Zhu: Writing – review & editing, Writing – original draft, Visualization, Validation, Software, Resources, Methodology, Investigation, Formal analysis, Data curation, Conceptualization.
Jian-Dong Huang: Writing – review & editing, Funding acquisition.
Ding-Hui Yang: Writing – review & editing, Supervision, Project administration, Funding acquisition.

Data availability

The data will be shared upon reasonable request to the corresponding author.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Stability analysis of the robust WRKDG methods

The plane wave analysis for the discontinuous Galerkin can be found in several studies (Hu et al., 1999; Ainsworth et al., 2006; De Basabe et al., 2008; De Basabe and Sen, 2010; He et al., 2014). Here, we present the growth matrices of the new methods on rectangular meshes.

For the coefficient-optimized weighted Runge-Kutta DG method, after spatial discretization, we set the growth matrix as follows:

$$W^{(n+1)} = \left(I + \frac{\Delta t}{2} (Q_1 + Q_1 Q_2) \right) W^{(n)}, \quad (\text{A-1})$$

$$Q_1 = S + (\eta_1 + \eta_2) r \Delta t S^2 + \eta_2 (r \Delta t)^2 S^3. \quad (\text{A-2})$$

$$Q_2 = I + (1 - 2r) \Delta t Q_1, \quad (\text{A-3})$$

where S denotes the matrix related to the spatial discretization after plane wave analysis. Its explicit form can be found in Hu et al. (1999) and He et al. (2020). We denote the growth matrix as A_1 , whose specific expression can be described as follows.

$$A_1 = I + \frac{\Delta t}{2} (Q_1 + Q_1 Q_2). \quad (\text{A-4})$$

For the Stepwise computation-reduced weighted Runge-Kutta, the growth matrix can be described as follows.

$$W^{(n+1)} = \left(I + \frac{\Delta t}{2} (Q_{3,1} + Q_{3,2} Q_4) \right) W^{(n)}. \quad (\text{A-5})$$

$$Q_{3,1} = S + \eta_1 r \Delta t S^2 + \eta_1 (r \Delta t)^2 S^3. \quad (\text{A-6})$$

$$Q_{3,2} = S + \eta_2 r \Delta t S^2. \quad (\text{A-7})$$

$$Q_4 = I + (1 - 2r) \Delta t Q_{3,1}. \quad (\text{A-8})$$

The growth matrix is given by A_2 .

$$A_2 = I + \frac{\Delta t}{2} (Q_{3,1} + Q_{3,2} Q_4). \quad (\text{A-9})$$

To determine the optimal parameters, we employ an exhaustive search method, the specific algorithm of which is analogous to that proposed by He et al. (2015) and detailed below. The optimal parameters selected through this optimization are presented in Table 3.

For all considered parameter combinations, we calculate the approximate CFL condition number of the proposed method. Specifically, parameters η_1 and η_2 are sampled over the interval $[0, 1]$ with a step size of 0.1, while parameter r is sampled over the same interval with a smaller step size of 0.01. The method for calculating the CFL condition number is given below.

Step 1: $\alpha = 0$.

Step 2: if $|A_1(kh, \alpha, \theta)| \leq 1, \forall kh \in [0, 2\pi], \forall \theta \in [0, 2\pi]$, then go to step 3.

else

exit and return α .

Step 3: $\alpha = \alpha + \Delta\alpha$.

For the stepwise computation-reduced weighted Runge-Kutta method, only the replacement of matrix A_1 with matrix A_2 in the

above algorithm is required. After calculating the CFL condition numbers for all parameters, we can select the near-optimal parameter sets; the corresponding CFL condition numbers are listed in Tables 1 and 2, and the selected parameter sets are presented in Table 3.

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